Journal of Economic Dynamics & Control 111 (2020) 103807

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Hindered growth

Moshe Elitzur[a,b,*], Scott Kaplan[c], David Zilberman[c]

[a] Department of Astronomy, Univ of California, Berkeley, CA 94720, USA
[b] Dept of Physics & Astronomy, Univ of Kentucky, Lexington, KY 40506, USA
[c] Dept of Agricultural & Resource Economics, Univ of California, Berkeley, CA 94720, USA

ARTICLE INFO

Article history:
Received 20 May 2019
Revised 12 November 2019
Accepted 13 November 2019
Available online 20 November 2019

JEL classification:
C1
C5
C6
N1
N3

Keywords:
Exponential growth
Logistic growth
Growth hindering
Time series

ABSTRACT

We develop a formalism to extract the exponential component from a growth process and describe the remainder with the optimal number of parameters. The method is demonstrated analyzing the time variation of Gross Domestic Product (GDP) and population in the US and UK, two nations with continuous data coverage going back more than 200 years. For each of the four datasets we find a successful description, with the deviation of long-term growth from a pure exponential requiring no more than a single free parameter; there is no significant gain from adding more parameters. We find persistent long-term growth patterns, consistent with Jones (1995) and showing directly from the data that population and GDP growth in different countries may follow different trajectories, illuminating their intrinsic differences.

1. Introduction

Growth impacts daily activities on many levels. Its prime indicator, the growth rate, plays a key role in decisions that determine quantities as diverse as the stock price of a public company and the prime rate of a national economy. The great interest in growth rates, by both policy makers and the general public, is reflected in the efforts invested in their measurement and dissemination. Every April and October the International Monetary Fund (IMF) releases detailed data and forecasts of economic and population growth by country.[1] The magazine *The Economist* updates twice daily economic and financial indicators, foremost among them GDP growth rate, for a large number of countries.[2]

For any quantity Q, such as a company's revenue, national GDP, etc., the growth rate $g = d\ln Q/dt$ is derived directly from the data without any modeling and is a predictor of the expected behavior of Q in the near future. However, this predictive power is diminished over periods sufficiently long that the growth rate itself may vary appreciably. For such predictions, detailed modeling of the actual time variation dQ/dt is necessary. In such modeling, key dynamic processes and their mutual interactions and interdependencies are identified and described mathematically. The model then accounts for all the

* Corresponding author.
E-mail addresses: moshe@g.uky.edu (M. Elitzur), scottkaplan@berkeley.edu (S. Kaplan), zilber11@berkeley.edu (D. Zilberman).

[1] https://www.imf.org/external/datamapper/datasets/WEO

[2] https://www.economist.com/economic-and-financial-indicators/2018/03/20/economic-and-financial-indicators

https://doi.org/10.1016/j.jedc.2019.103807
0165-1889/

relevant processes and interactions involving Q, and dQ/dt is determined from the balance between gains and losses. Such structural models have been developed by both economists (Becker et al., 1990; Jones, 2016; Solow, 1956) and demographers (Levin, 1976) to quantify processes that drive the growth of GDP and population. The literature on economic growth aims to decompose the contribution of factors like capital and labor, and identify the impact of processes of learning and buildup of human capital (Acemoglu, 2008; Mankiw et al., 1992). Academic demographers are interested in decomposing population growth dynamics into multiple processes like fertility and life expectancy (O'Neill et al., 2001). Estimation of dynamic systems that aims to identify causal relationships and incorporate multiple relevant factors applies complex autoregressive models (Akaike, 1969) or uses unobserved components models with permanent and stochastic deviations (Stock and Watson, 2007).

Detailed structural modeling is essential for insight into the inner workings of the studied systems. But for more limited purposes it is possible to take a simpler, purely data-driven approach: construct an analytic description of the long-term variation of Q without attempting to explain the specifics behind its growth. In this phenomenological approach the definition of g is inverted to become the growth equation

$$\frac{dQ}{dt} = gQ\,, \tag{1}$$

where g, too, is in general a function of t. With some suitable parametrization for g this equation can be solved in parametric form and the free parameters estimated from the best fit to the data, resulting in an analytic description for the time-dependent $Q(t)$. Such an approach might be feasible when underneath the volatile fluctuations common to growth rates, their long-term patterns follow reasonably smooth trajectories. While not exploring the reasons behind the growth, such analysis can still provide valuable information since it yields an analytic description with a few key parameters, making it possible to build a methodical, general classification scheme of growth patterns. Given a dataset, the challenge is to make an unbiased selection of the proper mathematical function to describe it analytically and determine the optimal number of parameters. This is what we set out to do here.

Our aim is to describe long term growth—although g may fluctuate significantly, and even become negative on occasion (during economic recessions, for example), our interest is only in depicting its underlying smooth variation. The key concepts behind the approach we take can be illustrated with a simple example. Consider a newly discovered desolate island into which apple seeds are introduced. Some seeds will sprout, apple trees will produce new seeds and the tree population will grow. Annual weather variations and an occasional infestation will cause fluctuations in the growth rate, but fundamentally it is determined by the island's climate, ground fertility, etc., and thus maintains a constant long-term average. Once the tree population has grown to the point that tree crowding becomes a significant factor, the growth rate begins to decline from its initial value. This simple example is illustrative of a general fundamental point: the growth of any quantity Q occurs within some environment, broadly defined as the collection of all the processes and system components that affect the growth of Q other than Q itself. As long as the growing Q is sufficiently small that its impact on the environment is negligible, the growth rate g is determined by intrinsic properties of the environment independent of the magnitude of Q itself. This rate is maintained until Q becomes sufficiently large that it significantly impacts the environment, at which point it also affects its own growth rate. In general, this causes the growth rate to decline and we refer to this effect as *growth hindering*—the growing quantity has become so large as to hinder its own growth. One interpretation of hindering is that there is an initial, "natural" unconstrained rate of growth, but as Q increases, its rate of growth is constrained and tends to diminish, consistent with the notion of decreasing marginal productivity. This is the phenomenon we wish to describe.

The critical magnitude for the onset of hindering and the specific reasons behind the effect vary from system to system. In the above example, the key parameter is the minimal area for viability of a single apple tree; hindering begins to affect the tree population growth when the average area per tree becomes a significant fraction of this parameter. But even without this knowledge we could discern the onset of hindering and determine the impact of the effect from time records of the island's apple tree population when the period covered by the data is sufficiently long. Similarly, given records of any quantity with sufficient time coverage one can detect, and describe, the effect of hindering from analysis of the data even when the exact specification of the environment and its interplay with the growing quantity are not understood.

We present here a general framework for fitting the long-term variation of Q with the minimal number of parameters, without preconditions about the nature of the growth, bounded or unbounded, and without prejudging the specific functional form of the fit to the data. To this end we formulate in Section 2 a general framework to describe hindering, and derive the general solution of the growth equation (Eq. 1) in parametric form. Our solution provides a generic description of growing quantities just as the Fourier series provides a generic description of periodic phenomena. It is based on a straightforward Taylor expansion so that finding the number of free parameters required to describe the time variation of Q becomes simply a matter of determining the number of terms that should be kept in a Taylor series. We emphasize *linear hindering* (Section 2.1), a single-parameter form for the time variation of the growth rate that allows unbounded growth, and derive also the time dependence of higher order terms (Section 2.2). We show that any unbounded growth can be described by a finite sum (Section 2.3) while infinite sums produce bounded growth, of which the logistic function is an example (Section 2.4). Thanks to the simplicity of this new framework we are able to successfully model over 200 years of GDP and population data for both the United States of America (Section 3.1) and the United Kingdom (Section 3.2). We show that no more than a single free parameter is needed to describe the deviations from pure exponential of the long-term variation of the growth of both population and GDP in these two nations; additional parameters make no significant difference. We summarize and discuss our results, including the benefits and limitations of our method, in Section 4.

2. A formalism for hindering

From its definition as $d\ln Q/dt$, the growth rate is a function of Q, itself a function of t. As long as the growing Q is sufficiently small that its impact on the environment is negligible, it grows at some unconstrained rate which we term the *unhindered growth rate* g_u; in the apple trees example (Section 1), g_u is the underlying growth rate for the tree population just as the very first seeds are introduced into the island. Since g_u is determined by intrinsic properties of the environment independent of the magnitude of Q itself, it can be defined from

$$g_\mathrm{u} = g(Q \to 0); \tag{2}$$

note that in practice g_u should be derived from the $Q = 0$ intercept of a trend line through the data to smooth out the effects of short-term fluctuations.

When hindering sets in, the impact of Q on its own growth rate becomes significant and g begins to decrease with Q so that $g(Q) \le g_\mathrm{u}$. Quantifying hindering requires a function that captures the essence of the effect—vanishing at $Q = 0$ and increasing with the rising impact of the growing Q on its own environment, i.e., increasing with a decreasing $g(Q)$. A simple such function is

$$f(Q) = \frac{g_\mathrm{u}}{g(Q)} - 1 \quad \text{so that} \quad g(Q) = \frac{g_\mathrm{u}}{1 + f(Q)}\,. \tag{3}$$

We refer to $f(Q)$ as the *hindering factor*. Thus defined, f varies in the same sense as the hindering effect: When $Q = 0$, f too is zero—the hindering factor vanishes in the absence of hindering, as it should; and since hindering (the impact of Q on its own growth rate) increases with Q and causes g to decrease, f increases with hindering so that $f'(Q)$ is expected to be ≥ 0. From its definition, $f > -1$. The range $-1 < f < 0$ corresponds to $g > g_\mathrm{u}$, i.e., accelerated growth, thus this growth pattern, too, can be studied with this approach. Decelerated growth (hindering) corresponds to $0 < g < g_\mathrm{u}$ and is mapped into $f > 0$. Here we deal only with non-accelerated growth, hence $f \ge 0$. The point $f = 1$, where $g = \frac{1}{2}g_\mathrm{u}$, marks a transition between two very different growth patterns: $f \ll 1$ corresponds to unhindered growth with a constant growth rate $g \simeq g_\mathrm{u}$ and Q rising exponentially with time; $f \gg 1$ yields hindered growth with $g(Q) \simeq g_\mathrm{u}/f(Q)$ $(\ll g_\mathrm{u})$ and a time variation for Q that depends on the specific functional form of f.

Inserting the definition of f into Eq. 1, the growth equation becomes

$$\ln Q + \int \frac{f(Q)}{Q}\, dQ = g_\mathrm{u} t. \tag{4}$$

The simple mathematical manipulations that lead from Eq. 1 to this result rely on the implicit assumption that Q is monotonically increasing ($dQ/dt > 0$), which ensures that Q is a single-valued function of t.[3] As a result, g can be properly considered a function of Q instead of t and this substitution leads to the separation of variables accomplished in Eq. 4: the function of Q on the left-hand-side varies linearly with time. The logarithmic term in this function corresponds to exponential growth with the constant growth rate g_u, therefore the integral involving the hindering factor f describes the deviations of the growth trajectory from a pure exponential. Constraining f to be positive ensures that the integral, too, is positive so that Q grows more slowly than exponential; that is, this term describes the hindering effect.

Expressing the growth rate in terms of the hindering factor (Eq. 3) leads to a full extraction of the exponential behavior, splitting the growth trajectory into two distinct components—exponential rise and the deviations from it[4]. *All the information about the deviations of the growth pattern from pure exponential is contained in the hindering factor f.* The actual form of the deviations depends on the specific hindering factor—every functional form of g defines a corresponding f, and vice versa. Other than its normalization $f(0) = 0$ and the requirement that it be positive, f is arbitrary. Assuming it to be reasonably behaved, it can be expanded in a power series with some coefficients a_k,

$$f = \sum_{k=1}^{\infty} a_k Q^k. \tag{5}$$

Inserting this series expansion in Eq. 4, straightforward term-by-term integration yields Q as a function of t in terms of the expansion coefficients a_k, producing the general solution of the growth equation in parametric form. Before writing the actual solution (see Section 2.3 below) we consider some specific cases, starting with the simplest hindering factor containing just a single expansion coefficient.

[3] This procedure cannot be repeated as is when Q is monotonically decreasing ($dQ/dt < 0$; contraction) because the limit $Q \to 0$ (Eq.2) cannot be taken in that case.

[4] Writing Q as $Q = e^{g_\mathrm{u} t} D$, the deviation D from exponential obeys $\ln D = -\int (f/Q)\, dQ$.

2.1. Linear hindering

Linear hindering is the case in which the series expansion of f is truncated right after its first order term. Introducing the *hindering parameter* $Q_h = 1/a_1$, the linear hindering factor is now

$$f = a_1 Q = \frac{Q}{Q_h}, \tag{6}$$

so that the growth rate is

$$g = \frac{g_u}{1 + Q/Q_h} \tag{7}$$

and Q_h is the point where g has declined to half its original, unhindered value. Unhindered growth, with $Q \propto \exp(g_u t)$, occurs while $Q \ll Q_h$ ($f \ll 1$). The hindered growth phase, $Q \gg Q_h$, has an entirely different behavior. During this stage $f \gg 1$ and $g \simeq g_u/f = g_u Q_h/Q$ — the growth rate is continuously decreasing while remaining finite for all finite values of Q, vanishing only in the limit $Q \rightarrow \infty$. Inserting this approximation in the growth equation (Eq. 1) yields the asymptotic behavior $Q \simeq Q_h g_u t$. Instead of exponentially, Q is now growing linearly with time, unbounded. The complete variation of Q with time is obtained from the integration in Eq. 4 with the linear form of f from Eq. 6. Denoting by Q_0 the value of Q at initial time t_0, the integration yields

$$\ln\frac{Q}{Q_0} + \frac{Q - Q_0}{Q_h} = g_u(t - t_0). \tag{8}$$

This result defines Q implicitly as a function of time in terms of the initial value Q_0 and the free parameters g_u and Q_h. An explicit analytic expression for $Q(t)$ is available in terms of the new mathematical function h_1, introduced in Appendix A (Eq. A.2; Fig. A.9 includes a plot of h_1), as $Q = Q_h h_1(g_u t; C)$ where C stands for the integration constant. Denote

$$x_0 = \ln\frac{Q_0}{Q_h} + \frac{Q_0}{Q_h} - 1. \tag{9}$$

Then from the expression for the inverse function of h_1 (Eq. A.4) it follows that $h_1(x_0) = Q_0/Q_h$, and the linear hindering solution in Eq. 8 becomes

$$Q(t; g_u, Q_h, Q_0) = Q_h\, h_1\big(g_u[t - t_0] + x_0\big). \tag{8$'$}$$

This equation describes linear-hindering growth in terms of the function h_1 just as Eq. 23 below describes logistic growth in terms of the exponential function.

The definition of Q through Eq. 8 (or 8′) involves the parameter Q_0 corresponding to the value of the function at one particular time, and thus is seemingly dependent on that specific choice. However, since t_0 is arbitrary it can be replaced by any other time, in particular the time t_h corresponding to $Q = Q_h$; from Eq. 8, t_h obeys

$$g_u(t_h - t_0) = 1 - \frac{Q_0}{Q_h} - \ln\frac{Q_0}{Q_h}. \tag{10}$$

Selecting for the integration constant the time t_h yields an alternative to Eq. 8 for the linear hindering solution:

$$\ln\frac{Q}{Q_h} + \frac{Q}{Q_h} - 1 = g_u(t - t_h). \tag{11}$$

Since t_h is the time when Q reaches the hindering parameter, it is an intrinsic property of the function, thus this form for Q explicitly demonstrates there is no arbitrariness in its definition. And since $h_1(0) = 1$ (Eq. A.1), the explicit solution in terms of the linear-hindering mathematical function h_1 is straightforward

$$Q(t; g_u, Q_h, t_h) = Q_h\, h_1(g_u[t - t_h]). \tag{11$'$}$$

Unhindered growth occurs while $t \ll t_h$; its characteristic exponential behavior follows directly from the full functional form of $Q(t)$ when only the logarithmic term is retained on the left-hand-side of Eq. 8 (or Eq. 11), neglecting the algebraic term which is smaller in this regime. Hindered growth occurs when $t \gg t_h$, and its linear behavior is obtained by dropping the logarithmic term on the left-hand-side.

Fig. 1 shows the variation of Q (top panel) and g (mid panel) with t for various values of the hindering parameter Q_h. Time is measured in units of the unhindered growth time[5]

$$T_u = \frac{1}{g_u}. \tag{12}$$

For convenience, g is normalized by g_u, and all Q, including Q_h, by Q_0. Short vertical lines on plots with $Q_h > Q_0$ mark the point $Q = Q_h$ where Q surpasses its hindering parameter. The unhindered growth phase, with its constant g and exponential Q, corresponds to $Q < Q_h$ and is easily discernible in the plots that have $Q_h > Q_0$. Every curve shows the hindered growth phase $Q > Q_h$ — the larger is Q_h, the longer it takes to reach it. The plots with $Q_h = 0.1Q_0$ present a system that has reached hindering long before the initial time shown. From Eq. 10, $g_u(t_h - t_0) = -11.3$ in this case, the system is deeply in the hindered phase for the entire duration of the displayed time.

[5] Also known as the e-folding time. The unhindered doubling time is $T_u \ln 2 = 0.69 T_u$.

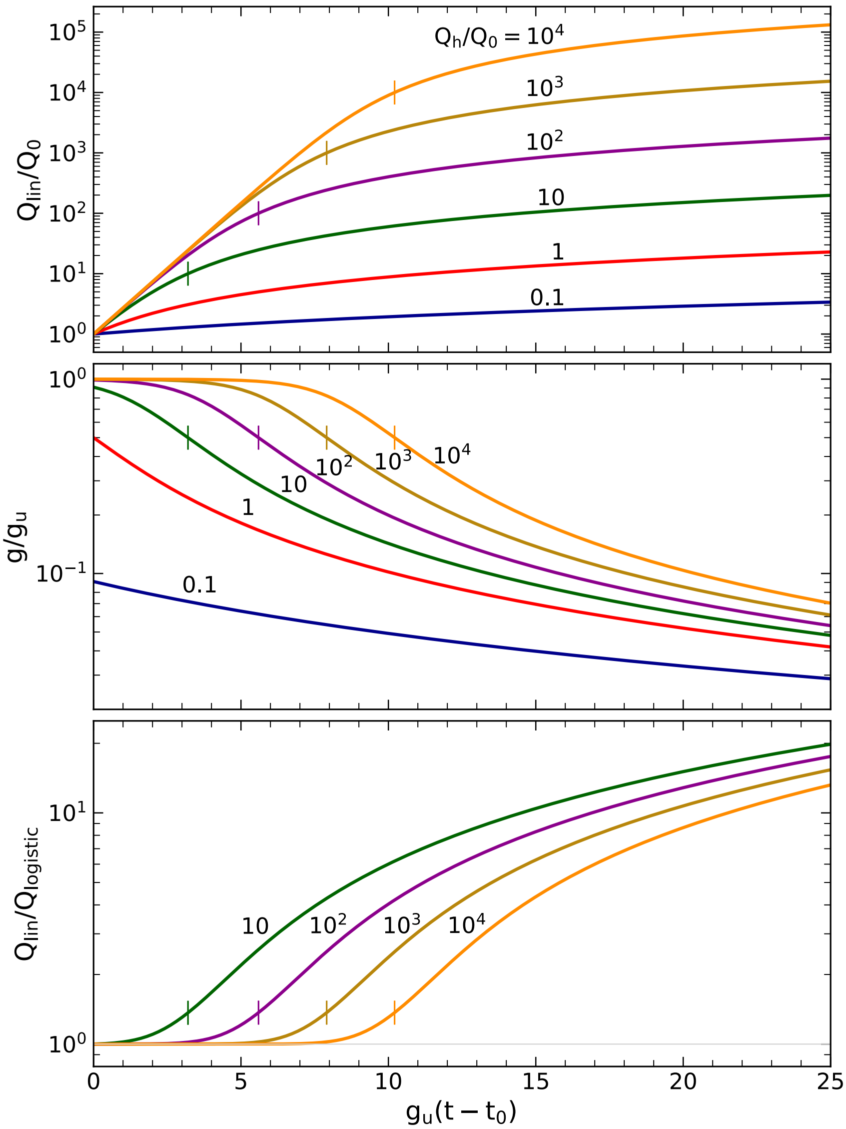

Fig. 1. Time variation of (*Top*) the linear hindering Q_{lin} (Eq. 8) and (*Middle*) its growth rate g (Eq. 7) for various values of the hindering parameter Q_h, as labeled. The plots use normalized quantities—Q is normalized by Q_0, g by g_u and t by T_u (Eq. 12). *Bottom*: For the cases when $Q_h > Q_0$, ratio of the linear hindering Q_{lin} and the corresponding logistic function $Q_{logistic}$ (Eq. 23) with the same g_u and Q_0, and $Q_C = Q_h$ (see Section 2.4 below). In all panels, a vertical mark on each plot with $Q_h/Q_0 > 1$ indicates the point where $Q_{lin} = Q_h$; the corresponding time t_h is given in Eq. 10. Note the log-linear scale of all plots.

2.1.1. Production vs amplification

The growth equation (Eq. 1) can be considered the description of a self-amplification process: g is the multiplier rate of Q so that the larger is Q the more it grows just because there is more of it. This interpretation is particularly useful when g is constant, explaining the mechanism behind the exponential rise of Q in that case. But this equation can also be viewed as the description of a production process of stock as a function of time, with gQ the production rate of Q. With g from Eq. 7, the linear-hindering production rate is

$$\frac{dQ}{dt} = g_u \frac{Q_h}{Q + Q_h} Q \simeq g_u \times \begin{cases} Q & \text{when } Q \ll Q_h \\ Q_h & \text{when } Q \gg Q_h. \end{cases} \tag{13}$$

The production rate increases linearly with Q as long as $f = Q/Q_h \ll 1$, slows down when $f \simeq 1$ and approaches saturation at its maximal value $g_u Q_h$ when $f \gg 1$. It is instructive to write the result as

$$\frac{dQ}{dt} = \epsilon \cdot \left.\frac{dQ}{dt}\right|_{max} \tag{14}$$

where

$$\frac{dQ}{dt}\bigg|_{\max} = g_u Q_h, \qquad \text{and } \epsilon = \frac{Q}{Q+Q_h} = \frac{f}{1+f}. \tag{15}$$

That is, $g_u Q_h$ is the maximal production rate and ϵ is the ratio of actual production rate to this maximum, namely, it is the production efficiency (also known as the capacity utilization rate)—when $\epsilon = 1$ the system grows utilizing its resources to the fullest. The hindering factor f and the efficiency ϵ vary together, only their ranges are different: ϵ varies from 0 to 1 when f varies from 0 to ∞; high hindering is high efficiency. The growth rate can be written similarly as

$$g = (1-\epsilon)\cdot g_u, \tag{16}$$

so that $1-\epsilon$ is the ratio of the actual growth rate to its maximum possible value of g_u, namely, it is the amplification efficiency. The efficiencies of production (ϵ) and amplification ($1-\epsilon$) are the mirror images of each other, explaining the different behavior of growth during the hindered ($f > 1,\ \epsilon > \frac{1}{2}$) and unhindered ($f < 1,\ \epsilon < \frac{1}{2}$) stages. Hindered growth, with a production efficiency $\epsilon \simeq 1$, is best understood as a production process with a constant production rate. As a result, the growth rate ($\simeq g_u/f$) is continuously decreasing simply because this constant ($g_u Q_h$) is becoming a smaller and smaller fraction of the continuously growing Q. On the other hand, during unhindered growth the amplification efficiency is $1-\epsilon \simeq 1$, thus the amplification approach is more useful since the amplification rate is constant (g_u) while the production efficiency $\epsilon(\simeq f)$ is low and varying. The growing Q has no impact on its own growth rate and thus is rising exponentially fast precisely because the production efficiency is low, with dQ/dt far below its full potential ($g_u Q_h$).

It is also useful to derive the variation rate of the growth rate itself:

$$\frac{dr}{dt} = -{g_u}^2\,\epsilon(1-\epsilon)^2, \qquad \text{so that } \frac{d\ln g}{dt} = -g_u\,\epsilon(1-\epsilon). \tag{17}$$

From this it follows that dg/dt is declining most rapidly when $\epsilon = \frac{1}{3},\ f = \frac{1}{2},\ Q = \frac{1}{2}Q_h$ and $g = \frac{2}{3}g_u$, while the most rapid decline of $d\ln g/dt$ occurs when $\epsilon = \frac{1}{2},\ f = 1,\ Q = Q_h$ and $g = \frac{1}{2}g_u$.

2.2. Higher order hindering

Replacing the linear hindering factor (Eq. 6) by the k-th order power-law $f = (Q/Q_h)^k$ leads to

$$\ln\frac{Q}{Q_h} + \frac{1}{k}\left[\left(\frac{Q}{Q_h}\right)^k - 1\right] = g_u(t-t_h), \tag{18}$$

the equivalent of the solution in Eq. 11. With the aid of the mathematical hindering function h_k, introduced in Appendix A (Eq. A.6) and plotted in the top panel of Fig. A8 for a few values of k, this solution can be written in the explicit form $Q = Q_h\, h_k(g_u[t-t_h])$; Eq. 11′ is the private case $k=1$. Irrespective of the value of k, the unhindered domain ($t \ll t_h$) has $Q \ll Q_h$ and Q increases exponentially with time. But the hindered growth phase does depend on k: the asymptotic behavior when $Q \gg Q_h$ (i.e., $t \gg t_h$) is $Q \sim t^{1/k}$; hindered growth becomes flatter with increasing k. The production rate of Q is now

$$\frac{dQ}{dt} = g_u Q_h \frac{Q/Q_h}{1+(Q/Q_h)^k} \simeq g_u Q_h \times \begin{cases} \dfrac{Q}{Q_h} & \text{when } \dfrac{Q}{Q_h} \ll 1, \\ \left(\dfrac{Q_h}{Q}\right)^{k-1} & \text{when } \dfrac{Q}{Q_h} \gg 1. \end{cases} \tag{19}$$

Again, the unhindered domain ($Q/Q_h \ll 1$) exhibits a k-independent behavior $dQ/dt \simeq g_u Q$, a rising derivative. But in the hindered domain, $Q/Q_h \gg 1$, dQ/dt exhibits a k-dependent behavior, with $k=1$ standing out as the only hindering mode with a constant derivative. All other values of k result in a decreasing derivative when $Q/Q_h \gg 1$, therefore the derivative has a maximum around $Q \sim Q_h$; that is, the production rate increases linearly to a maximum, after which it decreases in proportion to $1/Q^{k-1}$. The maximum of the derivative of the mathematical h_k is derived in Appendix A (see Eq. A.7). From that it follows that when $k > 1$, the derivative peak value is $dQ/dt\big|_{\max} = g_u Q_h\,\frac{1}{k}(k-1)^{\frac{k-1}{k}}$ and it is reached at $t = t_h + x_k T_u$, where T_u is from Eq. 12 and x_k is from Eq. A.7; at this point $Q = Q_h/(k-1)^{1/k}$. In particular, quadratic hindering ($k=2$) has $dQ/dt\big|_{\max} = \frac{1}{2}g_u Q_h$, reached at $t = t_h$ where $Q = Q_h$. The bottom panel of Fig. A8 shows h'_k, the derivative of the mathematical k-th order hindering function, for $k \le 5$. The fundamental differences between $k=1$ and $k>1$ stand out.

2.3. Solution of the growth equation

The most general hindering factor f involves a combination of power laws in a Taylor series expansion (Eq. 5). Assuming the linear term to be present, i.e., $a_1 \neq 0$, the relative importance of higher order terms can be assessed by switching to $\alpha_k = a_k Q_h^k$ (namely, $a_k = \alpha_k a_1^k$); obviously $\alpha_1 = 1$. Then the hindering factor becomes

$$f(Q;\{\alpha_k\}) = \sum_{k=1}^{\infty} \alpha_k \left(\frac{Q}{Q_h}\right)^k, \qquad \alpha_1 = 1 \tag{20}$$

where $\{\alpha_k\}$ denotes the set of expansion coefficients. For each $k > 1$, the coefficient α_k sets the significance level of the k-th order term relative to the linear one. For example, quadratic hindering is negligible so long as $\alpha_2\, Q/Q_{\mathrm{h}} \ll 1$ (see also Eq. 28, below, and subsequent discussion; Appendix A provides some additional properties of mixtures of linear and quadratic hindering). With this expression for f, straightforward integration of Eq. 4 yields the equivalent of Eq. 11:

$$\ln\frac{Q}{Q_{\mathrm{h}}} + \sum_{k=1}^{\infty}\frac{1}{k}\alpha_k\left[\left(\frac{Q}{Q_{\mathrm{h}}}\right)^k - 1\right] = g_{\mathrm{u}}(t - t_{\mathrm{h}}), \tag{21}$$

with a similar replacement for Eq. 8. As with the case of linear hindering, the solution can be written in the explicit form

$$Q = Q_{\mathrm{h}}\, h(g_{\mathrm{u}}[t - t_{\mathrm{h}}]; \{\alpha_k\}) \tag{21$'$}$$

where h is the mathematical function introduced in Eq. A.9 of Appendix A[6] *This is the general solution of the growth equation (Eq.1) in parametric form*. With a suitable selection of $\{\alpha_k\}$ ($k \ge 2$), any growth pattern can be described by Eq.21 (or 21′) provided $dQ/dt > 0$; just as the Fourier series provides a generic description of periodic phenomena, this is a generic description of growth. The mathematical function h is controlled by the set of dimensionless free parameters $\{\alpha_k\}$. It is transformed into the actual solution with the aid of g_{u} and Q_{h}, free parameters that set the relevant dimensional scales for t and Q, respectively, and t_{h}, related to the boundary condition.[7] When $a_1 = 0$, the procedure outlined here can be repeated by replacing a_1 with the first non-vanishing coefficient. If that coefficient is a_p, introduce $Q_{\mathrm{h}} = (1/a_p)^{1/p}$ and then $\alpha_k = a_k Q_{\mathrm{h}}^k$ for $k \ge p$ as above.

In the limit $Q/Q_{\mathrm{h}} \to \infty$, f increases without a bound. If f is dominated by its k-th order term, the asymptotic behavior of Q is ~ $t^{1/k}$ as shown above (Section 2.2). Larger values of k provide flatter growth—the higher order the hindering, the flatter the increase of Q with time. Therefore, when the hindering factor contains a finite sum of K terms with monotonically decreasing α_k, all $\ll 1$, the time variation of Q is as follows: After an initial exponential rise, when Q surpasses Q_{h} its growth becomes proportional to t. Once the 2nd-order term starts dominating, the behavior flattens to $Q \propto t^{1/2}$, then flattens further to $t^{1/3}$ and so on. Finally, when f is dominated by the last term in its series expansion the time variation settles into $Q \propto t^{1/K}$, unbounded growth that continues indefinitely. *Hindering factors with a finite sum of expansion terms describe unbounded growth*. In the limit $K \to \infty$, the $t^{1/K}$ behavior approaches a constant that sets the upper limit on Q. *Bounded growth corresponds to hindering factors whose series expansion contains an infinite sum of terms*. We now describe one particular example of bounded growth.

2.4. The logistic hindering

The logistic growth function reflects various processes, with imitation and diffusion among them. It is used to analyze many phenomena, including population dynamics (Kingsland, 1982) and diffusion of technology (Griliches, 1957). The logistic growth rate is

$$g = g_{\mathrm{u}}\left(1 - \frac{Q}{Q_{\mathrm{C}}}\right), \tag{22}$$

where g_{u} and Q_{C} (the carrying capacity) are constants. Since g vanishes when $Q = Q_{\mathrm{C}}$, the logistic describes bounded growth with Q_{C} the absolute upper limit on Q. Denoting by Q_0 ($< Q_{\mathrm{C}}$) the value of Q at initial time t_0, the solution of the logistic growth equation is

$$Q = \frac{Q_{\mathrm{C}}}{1 + (Q_{\mathrm{C}}/Q_0 - 1)e^{-g_{\mathrm{u}}(t - t_0)}}. \tag{23}$$

This function makes an S-shaped exponential transition from Q_0, its initial value, to Q_{C}, the final one. With T_{u} the unhindered growth time (eq. 12), the transition occurs over a period with duration of a few T_{u} and is centered on the time $t_1 = t_0 + T_{\mathrm{u}}\ln(Q_{\mathrm{C}}/Q_0 - 1)$, where $Q = \frac{1}{2}Q_{\mathrm{C}}$ and $g = \frac{1}{2}g_{\mathrm{u}}$. When $t \ll t_1$, $Q \simeq Q_0$ and when $t \gg t_1$, $Q \simeq Q_{\mathrm{C}}$. The production rate

$$\frac{dQ}{dt} = g_{\mathrm{u}}Q_{\mathrm{C}}\cdot\frac{Q}{Q_{\mathrm{C}}}\left(1 - \frac{Q}{Q_{\mathrm{C}}}\right) \tag{24}$$

reaches a maximum of $\frac{1}{4}g_{\mathrm{u}}Q_{\mathrm{C}}$ at time t_1 and vanishes when $Q = Q_{\mathrm{C}}$.

The logistic growth rate displays the hindering effect. The corresponding hindering factor (Eq. 3) is

$$f = \frac{Q}{Q_{\mathrm{C}} - Q} = \sum_{k=1}^{\infty}\left(\frac{Q}{Q_{\mathrm{C}}}\right)^k. \tag{25}$$

As expected for growth bounded by an upper limit, the series expansion of f extends to infinity (Section 2.2). The first expansion term, Q/Q_{C}, is the linear hindering factor with $Q_{\mathrm{h}} = Q_{\mathrm{C}}$ (Eq. 6), thus the logistic Q_{logistic} can be directly compared

[6] Fig. A9 plots the function $h(x;\, \alpha_2)$, where the set $\{\alpha_k\}$ is truncated after $k = 2$, for various α_2.

[7] Additional discussion of the transformation from a purely mathematical function to the actual solution for Q is provided in the second paragraph of Appendix A for the linear hindering case.

with the linear hindering $Q_{\rm lin}$ that has $Q_{\rm h} = Q_{\rm C}$ and the same $g_{\rm u}$ and Q_0. As long as $Q \ll Q_{\rm C}$, hindering has a negligible effect, both functions grow exponentially and the ratio $\rho = Q_{\rm lin}/Q_{\rm logistic}$ is 1. When Q increases, hindering slows down the rate of growth as the hindering factor f is increasing. The growth rate declines to $\frac{1}{2}g_{\rm u}$ when $f = 1$, a point reached when $Q_{\rm logistic} = \frac{1}{2}Q_{\rm C}$ but only when $Q_{\rm lin} = Q_{\rm C}$; that is, the hindering effect is much stronger for the logistic than for linear hindering. The reason is that the growth of $Q_{\rm lin}$ slows down when the impact of $Q/Q_{\rm C}$ is significant, and since the ratio of every pair of successive terms in the logistic series expansion (Eq. 25) is the same, $Q/Q_{\rm C}$, all the series terms are contributing to its hindering factor. As a result, $Q_{\rm logistic}$ is affected by hindering more strongly than $Q_{\rm lin}$, its growth rate is reduced by a larger amount and the ratio ρ is rising. At time $t_{\rm h}$, when the linear-hindering solution reaches $Q_{\rm lin} = Q_{\rm C}$ (Eq. 10), ρ reaches the value $1 + 1/e = 1.37$ for $Q_{\rm h}/Q_0 \gg 1$. The ratio keeps rising with time indefinitely since $Q_{\rm lin}$ grows without bound while $Q_{\rm logistic}$ is bounded.

The bottom panel of Fig. 1 shows $Q_{\rm lin}/Q_{\rm logistic}$ as a function of time for the relevant values of $Q_{\rm h}$ (those obeying $Q_{\rm h} > Q_0$). A short marker on each curve indicates the point where $Q_{\rm lin} = Q_{\rm h}$. As is evident from the figure, the ratio ρ starts deviating from unity at considerably earlier stages of growth, growing without bound as time increases.

3. Data analysis

Given a set of data points $q_0, q_1 \ldots q_N$ at times $t_0, t_1 \ldots t_N$ that describe a growth process, we determine the best-fitting parameters $g_{\rm u}$, $Q_{\rm h}$, $t_{\rm h}$ and $\{\alpha_k\}$ ($k \geq 2$) of the general growth solution (Eq. 21′) by minimizing the sum of squares of residuals of the data points q_i from the model points $Q_i = Q(t_i; g_{\rm u}, Q_{\rm h}, t_{\rm h}, \{\alpha_k\})$. Because of the large dynamic range spanned by typical records of GDP and population, we give all data points equal relative weights ($\sigma_i \propto q_i$) so that the minimization is performed on the sum $\sum_i (Q_i/q_i - 1)^2$. The power series part of the solution can be terminated when the ratio of successive terms in the series expansion of f (Eq. 20) that underpins the solution falls below a prescribed level of accuracy—the marginal contribution of the truncated tail is then negligible. With N data points and a desired level of accuracy δ, the Taylor series for f can be truncated after K terms, where K is determined from

$$\frac{\alpha_{K+1}}{\alpha_K}\frac{Q_N}{Q_{\rm h}} < \delta. \tag{26}$$

Since Q_N is the largest value of any model point, this condition ensures that the accuracy criterion is met by all other points as well. In addition to the finite-series parametrization, we also find the best-fitting logistic model (Eq. 23). Comparing residuals for this and the finite-series hindering solution, we determine the overall best-fitting model for the given dataset. It may be noted that the optimal number of parameters for the best-fitting model can also be determined in an alternate, purely statistical approach based on the F-test, which we discuss in Appendix B. The two approaches produced the same results in our analysis.

In addition to the fitting itself, our modeling process involves both pre- and post-modeling steps. Pre-modeling analysis is done to ensure that the data obey the assumptions underlying our general solution. Our primary assumption is $dQ/dt > 0$, i.e., $g > 0$. To test it we compute the growth rates from a finite-differences calculation of the pairs $(q_0, t_0), (q_1, t_1) \ldots$ and fit the resulting sequence of $g_0, g_1 \ldots$ with a smoothing spline. When the spline is positive, the assumption is verified. In addition, our formalism is geared toward decelerated, rather than accelerated, growth, namely $dg/dt \leq 0$. Before embarking on actual fitting, we determine whether that is the case directly from the data, performing the Mann-Kendall (hereafter MK) test on the g-series. This non-parametric test determines whether or not there is a monotonic trend in a given dataset without assuming the data points to be distributed according to any specific rule (in particular, there is no requirement that they be normally distributed); Kocsis et al. (2017) provide a detailed description of the MK-test, including an extensive bibliography of recent applications. The null hypothesis (H_0) is no trend in the time series, in which case the test's normalized statistic Z is distributed according to the normal distribution with a zero mean and unity standard deviation, $(1/\sqrt{2\pi})\exp(-Z^2/2)$. Positive (negative) Z indicates an increasing (decreasing) trend, the most extreme values that can be produced by a series with N terms are $\pm[\frac{1}{2}N(N-1) - 1]/\sqrt{N(N-1)(2N+5)/18}$. Attempting to determine the presence of hindering, we test the null hypothesis against the alternative hypothesis (H_a) that there is a downward monotonic trend ($Z < 0$) in a one-tailed test. The presence of hindering is established at the 99% confidence level when $Z < -2.5$. In that case, self-consistency of the modeling requires that the hindering factor exceed the prescribed level of accuracy, i.e., $f(Q_N) > \delta$.

Finally, we perform post-fitting analysis to assess the quality of the fit. This is done with the coefficient of determination

$$R^2 = 1 - \frac{\sum_i (q_i - Q_i)^2}{\sum_i (q_i - \bar{q})^2}, \tag{27}$$

where $\bar{q}$ is the average of q_i. This coefficient is a measure of the percentage variability of the dependent variable that has been accounted for by the model (Draper and Smith, 1998).

To test the robustness of our fitting algorithm we used the linear hindering solution (Eq. 8) with input parameters $g_{\rm u}$ = 5% per year and $Q_{\rm h}/Q_0 = 20$ to generate a time series spanning 500 years of growth from $Q_0 = 5$ million to 1.91 billion. Data analysis of this series with our fitting algorithm properly recovered the underlying linear hindering model with the precise values of the input parameters. Next we ran 1000 simulations of Gaussian noise added to every model point. Adding such noise to the quantity itself, rather than its growth rate, yields wildly fluctuating growth. With noise amplitude of 10%, twice as large as g_u, expansion flips to contraction every fourth year, on average, producing a mock time series that is much more erratic than the actual GDP and population datasets considered below (Sections 3.1, 3.2). While the MK-test was barely able to detect a downward trend in the highly volatile growth rates of this mock dataset, our fitting algorithm returned a linear hindering model with all parameters within 2%, on average, of the underlying model input values. Increasing the noise level, at an amplitude of 15% the MK-test did not recognize a downward trend in the growth rates in most simulations, yet our fitting algorithm still returned a linear hindering solution with model parameters within 5%, on average, of the input values. The algorithm failed to return a solution only when the noise amplitude reached 30%, six times larger than g_u; at this point the growth alternated almost annually between contraction and expansion, with g varying from -25% to 35% per year, typically.

Our fitting algorithm is based on a general solution of the equation of growth (Eq. 21) and thus is applicable for any dataset that obeys the assumptions underpinning this solution. The data points need not be equally spaced in time; the only requirement is that they sample a sufficiently lengthy period to uncover the underlying growth trend and enable a meaningful determination of the expansion coefficients. As is evident from Eq. 4, the time scale for evolution of the hindering effect is the unhindered growth time T_u (Eq. 12). An unhindered growth rate of 5% per year gives $T_u = 20$ years, at $g_u =$ 1% per year T_u becomes 100 years. To allow for a meaningful determination of the hindering effect, the data must contain a few growth times and span at least 100 years or so. The longest data sets that we found with continuous, annual coverage of population and GDP involve the US and UK. We now proceed with data analysis for these two nations.

3.1. US historical data

Johnston and Williamson (2019) provide tabulations of the annual US population and GDP, in 2012 dollars, from 1790–2018. With 229 data points, each tabulation provides the longest continuous data series we found in the literature. We use these data for hindered growth modeling.

3.1.1. US Population

From 3.9 million in 1790, the US population increased by a factor of 83 to 327 million in 2018. The time variations of the population and the associated growth rate are shown in the top and bottom panels, respectively, of Fig. 2. While the population is increasing in a fairly smooth manner, the growth rate is decreasing with rather large fluctuations. The underlying long-term variation of the growth rate can be captured by smoothing, which we performed with the cubic spline shown in the figure's bottom panel. The downward trend of the growth rate stands out in the smooth spline and is confirmed by the MK-test. The test yields a statistic $Z = -16.5$, more than 16 standard deviations from the null hypothesis expected mean and close to the most extreme value of $Z = -22.5$ that can be produced by a monotonically decreasing series of the same length. The corresponding p-value is $1.28 \cdot 10^{-61}$, thus the alternative hypothesis of a downward monotonic trend is accepted with great confidence—the probability for a false positive is vanishingly small. The MK-test overwhelmingly verifies the impact of hindering on US population growth.

With the hindering impact decisively established, the hindering factor f (Eq. 20) must be included in the modeling, and it has to contain at a minimum the linear term. To determine where the series expansion can be truncated (Eq. 26), at least one more term must be included, thus the modeling has to start with the general quadratic form of the hindering factor

$$f(Q) = \frac{Q}{Q_h}\left(1 + \alpha_2 \frac{Q}{Q_h}\right). \tag{28}$$

The best fitting parameters are $Q_h = 9.59 \cdot 10^7$ and $\alpha_2 = 9.32 \cdot 10^{-24}$, yielding $Q_N = 3.21 \cdot 10^8$ as the model result for 2018, the last year of the time series. Thus the linear term in the expansion of $f(Q_N)$ is $Q_N/Q_h = 3.34$ and the ratio of quadratic-to-linear terms is $\alpha_2 Q_N/Q_h = 3.12 \cdot 10^{-23}$. The quadratic term is entirely negligible, linear hindering suffices to describe the entire data set, with the hindering factor reaching a maximum of $f(Q_N) = 3.34$.

Fig. 2 shows the results for the best-fitting linear-hindering model with the listed parameters. As seen in the top panel, for the most part the data and the model can hardly be distinguished from each other. The quality of the fit is further demonstrated by its coefficient of determination $R^2 = 0.9978$ (Eq. 27)[8], which shows that almost all of the variability of the dependent variable has been accounted for. Additionally, the ratio of model to data, displayed in the mid-panel, shows only mild fluctuations around unity with an average amplitude of 2.5%. Similarly, the bottom panel shows that other than the edges of the time series, the growth rates from the model and the smoothing spline are virtually identical. Linear hindering properly reproduces the underlying long-term variation of the US population growth.

Linear hindering adequately fits the US population data. To demonstrate the impact of hindering, Fig. 2 shows also the pure exponential arising from removal of the hindering effect, $Q_0 \exp(x)$ where Q_0 is the hindered model result for the 1790 population (3.8 million) and x is the top x-axis of the figure. Each panel displays the results of this 'de-hindered' model. Initially identical to the hindered model, after 30–40 years it starts showing systematic deviations away from the data that increase with time, predicting a 2018 US population that is 27 times its actual value. Adding a single term to account for the time variation of the growth rate, linear hindering (Eq. 6) successfully rectifies this systematic drift away from the data.

[8] Taking both model and data in log yields a slightly larger $R^2 = 0.9994$.

With hindering affecting roughly 80% of the time series, it is noteworthy that the time variation of such a lengthy portion of the data is fitted successfully with just a single free parameter.

The linear hindering model shows that the US population surpassed its hindering parameter $Q_h = 95.9$ million in 1913, a point marked on the plots of the population and the growth rate, as well as the bottom x-axis of Fig. 2. By the year 2018 the linear term on the left hand side of Eq. 11 is exceeding the logarithmic one by a factor of almost 3. Having moved into the hindered growth phase, the US population growth is dominated by linear increase with time, adding roughly the same amount of 3.24 million people per year (see also Eq. 15). For the year 2050 the model predicts a US population of 400 million and a growth rate of 0.65% per year. To gauge the robustness of these predictions we cut the time series tail at years before 2018, fitted linear-hindering models to the truncated data and compared the model predictions for 2050 from these partial data sets with those of the full time series. Removing points from the series tail down to the year 1964 yields predictions for 2050 that are always within 5% of those by the full data set. And removing data points all the way down to 1923, every prediction still falls within 10% of that by the full time series. That is, the predictions for the year 2050 change by less than 10% even with as much as 40% of the data discarded.

We have also modeled the data with the logistic function (Eq. 23), and its best-fit solution produced an error estimate 6-times larger than the linear hindering model. The carrying capacity of the best-fitting logistic model, predicted to be the upper limit to the US population, is $Q_C = 308$ million people. This limit has been surpassed already in 2009, and the model prediction for 2050 of 294 million is lower than the 2005 US population. Thus the logistic model is rejected entirely on its own, even without the comparison of its fit quality with linear hindering. The reason this model can be eliminated so decisively is that the data provide meaningful constraints on the functional form of the hindering factor f since a large fraction of the time series is affected by hindering, as is evident from the decline of the growth rate in the bottom panel of Fig. 2. It is important to note that if the data were limited to the period, say, 1790–1995, the case against the logistic model would still be strong but not as decisive. It is the final ~ 20 years of data that make for an airtight case.

3.1.2. US GDP

Even with its limitations, GDP has been a major economic indicator for many decades. The US GDP has grown from \$4.57 billion in 1790 to \$18.6 trillion in 2018 (all \$ signs are in 2012 dollars), a 4,066-fold increase. With $Z = -3.37$, the MK-test establishes the presence of a downward monotonic trend in the growth rate with a p-value of $3.79 \cdot 10^{-4}$. While the evidence for hindering is decisive, it is not as overwhelming as for the US population, indicating that GDP growth might be at earlier stages of hindering. The best-fitting parameters for modeling with the quadratic hindering form of f (Eq. 28) are $Q_h = \$3.23 \cdot 10^{13}$ and $\alpha_2 = 7.2 \cdot 10^{-15}$. The model result for 2018 is $Q_N = \$1.97 \cdot 10^{13}$, so the first term of the hindering factor is $Q_N/Q_h = 0.61$ and the ratio of second-to-first is $\alpha_2\, Q_N/Q_h = 4.39 \cdot 10^{-15}$. As was the case with the population, the quadratic term in the hindering factor is entirely negligible, linear hindering suffices to describe the entire data set. However, with $f(Q_N) = 0.61$, the US GDP is at a much earlier stage of hindering than the population, in accordance with the MK-test results.

The recent onset of hindering stands out in Fig. 3, which shows the data and modeling results. As is evident from the spline smoothing of the data in the bottom panel, the GDP underlying growth rate remained constant, equal to its unhindered value of 3.83% per year, until around 1960. Afterwards the hindering effect manifests itself in the steady decline of the growth rate and the systematic divergence of the data away from the 'de-hindered' counterpart to the model; without the hindering, this counterpart over-predicts the 2018 GDP by a factor of 2. The hindering parameter is $Q_h = \$32.3$ trillion, and it will be surpassed only in 2042. Yet although the 2018 GDP is only 58% of Q_h, the most rapid decline of GDP growth rate has already occurred in 2012 (see Eq. 17), and in 2018 the linear term on the left hand side of Eq. 11 has become 96% of the logarithmic one. Even with its recent onset, hindering is already making a significant impact, which is evident in the figure and is decisively validated by the MK-test. The coefficient of determination is $R^2 = 0.9969$.

The time variation of US GDP fluctuates considerably more than the population; as is evident from the two mid panels in Figs. 2 and 3, the GDP fluctuates more frequently and with much larger amplitudes. However, these fluctuations dampened considerably after World War II. While the overall average deviation of GDP model from data is 8.96%, the average deviation during 1950–2018 was only 3.49%, with the largest deviation occurring in 2000 when the GDP exceeded the model prediction by 7.98%. Indeed, during the post-World War II period, the model and data are virtually identical in the top panel of Fig. 3 (see also middle panel), and the model and spline smoothing can be barely distinguished from each other in the bottom panel. For the year 2050 the linear hindering model predicts a GDP of \$37.4 trillion and an underlying growth rate of 1.77% per year. Truncating the tail of the time series all the way to 1993 and modeling with only partial data leads to predictions for 2050 that stay within 10% of those for the full data set.

Unlike the population case, logistic hindering provides a viable alternative model for the US GDP. The relatively short fraction of the time series significantly affected by hindering prevents a meaningful choice between the logistic- and linear-hindering. The error estimator[9] is virtually the same for both—3.95 for linear hindering vs 3.98 for logistic hindering. The best-fitting parameters for the logistic model (Eq. 23) are $g_u = 3.81\%$ per year, almost the same as the 3.83% per year for the linear hindering, and a carrying capacity of $Q_C = \$42.5$ trillion, implying that the US GDP will never exceed this upper bound. Therefore the logistic model could be eliminated if and when the US GDP surpassed \$42.5 trillion; the linear hindering model predicts this to happen in 2057, thirty nine years after the end of the time series. But a meaningful selec-

[9] $\sum(\text{model}/\text{data} - 1)^2$

tion might still be possible at an earlier time because of the slow, steady divergence of the two models. Beginning in 2010, the logistic has been falling systematically below linear hindering (for the explanation of such departure see the discussion in Section 2.4 and bottom panel of Fig. 1). The logistic model prediction for 2018 is 1.9% below that of linear hindering, and this deviation is growing continuously to 5% in 2026, 10% in 2037 and 15% in 2046. As noted above, thanks to the reduced fluctuations of GDP, the average deviation of model from data for the past 60 years is less than 4%. By the year 2026, the difference between the logistic and linear hindering models is predicted to exceed this average deviation. Thanks to this systematic divergence of the two models, a meaningful choice might be possible with roughly 10 more years of data. The logistic predictions for the year 2050 are GDP of $31 trillion and annual growth rate of 1.03%; these predictions are, respectively, 83% and 58% of those by the linear hindering model.

3.1.3. US GDP per capita

GDP per capita (GDPPC) is a general indicator of standard of living for a citizen, and adjusts for the size of a country's population when considering a measure of welfare for its citizens. The US GDPPC has grown from $1,163 in 1790 (2012 dollars) to $56,718 in 2018, increasing by a factor of 49; since the US population increased by a factor of 83 during that period, its contribution to the overall rise of GDP was 1.7 times larger.

GDP per capita is the result of two phenomena we have already analyzed. Following the same steps as taken in the analysis of the data themselves, we model the GDPPC with the ratio of the two models for the US GDP and population. Fig. 4 shows the results. As with the previous cases, the model captures successfully the long term variation of GDPPC: the average deviation of model from data is 8.09%, and except for the edges of the time series, the bottom panel shows that the model growth rate can be barely distinguished from the spline smoothing of the data. The coefficient of determination is $R^2 = 0.9938$. Similar to the GDP and population, the US GDPPC has been steadily rising with time. But in contrast with the former two, its spline-smoothed growth rate has been initially *increasing* too for more than 160 years, slowly decreasing after reaching a peak of 2.31% per year in 1953. Indeed, it is straightforward to show that the ratio of two linear hindering functions does not yield another linear hindering function, except for special circumstances (no hindering, or the trivial case when the two functions are proportional to each other). Therefore, there is no a priori expectation for the growth rate of GDPPC to follow any particular functional form.

The actual behavior of the GDPPC growth rate can be qualitatively understood from some simple considerations. Denote the growth rates of GDP, population and GDPPC by, respectively, $g_{\rm G}$, $g_{\rm P}$ and $g_{\rm C}$. Because GDPPC is the ratio of GDP to population, $g_{\rm C} = g_{\rm G} - g_{\rm P}$. The unhindered growth rate is larger for the US GDP than for its population, therefore $g_{\rm C}$ started positive. Had $g_{\rm G}$ and $g_{\rm P}$ remained constant, $g_{\rm C}$ too would have been constant. In actuality, both $g_{\rm G}$ and $g_{\rm P}$ decline because of hindering, but $g_{\rm P}$ declines more rapidly since the population has already surpassed its hindering parameter in 1913 while the GDP has not, and will be reaching that point only in 2042. As a result, $g_{\rm G}$ will continue to be larger than $g_{\rm P}$, and $g_{\rm C}$ will stay positive so long as the models with the parameters determined here remain applicable. However, the time variation of $g_{\rm C}$ depends on the difference between the decline rates of $g_{\rm G}$ and $g_{\rm C}$. As shown above (see Eq. 17 and subsequent discussion), the time derivative of linear-hindering growth rate is positive when $g \le \frac{2}{3} g_{\rm u}$ and negative thereafter. For the population, the growth rate has reached its peak decline rate in 1878, and the decline has been slowing down ever since. On the other hand, the decline of $g_{\rm G}$ has been accelerating until 2013. As a result, the rates of change of $g_{\rm G}$ and $g_{\rm P}$ became equal to each other in 1953, resulting in a maximum for $g_{\rm C}$[10] The model predicts that in 2050 the US GDP per capita will be $93,520 and have a growth rate of 1.12% per year.

3.2. UK historical data

Thomas and Williamson (2019) provide tabulations of the annual UK population and GDP in 2013 pounds, which we converted to 2012 dollars[11] Their *Consistent Series* is the series from 1700 to 2018 that is consistent in terms of the geographical area it is measuring, which includes England, Wales, Scotland and Northern Ireland. Because of lack of available data, prior to 1801 Northern Ireland is assumed to be a constant share of the UK GDP. To reduce uncertainty we discard that portion of the data, restricting our analysis to 1801–2018, the period fully covered by actual data.

3.2.1. UK Population

Unlike the other quantities discussed here, the UK population has increased by only a factor of 5.7 in 218 years, growing from 11.7 million in 1801 to 66.5 million in 2018. Even with this relatively small variation, an overall decrease in the growth rate is evident in Fig. 5. The MK-test yields $Z = -13.5$ (the most extreme value possible is $Z = -21.97$) and a p-value of

[10] Assume that both $g_{\rm G}$ and $g_{\rm P}$ follow the linear hindering relation in Eq. 7, with $g_{\rm u,G}$ and $g_{\rm u,P}$ the corresponding unhindered growth rates, and denote by $\epsilon_{\rm G}$ and $\epsilon_{\rm P}$ the respective efficiency factors (Eq. 15). From Eq. 17 it follows that $g_{\rm C}$ $(= g_{\rm G} - g_{\rm P})$ has a maximum that can be found from the solution of

$$\frac{\epsilon_P(1-\epsilon_P)^2}{\epsilon_G(1-\epsilon_G)^2} = \frac{g^2_{\rm u,G}}{g^2_{\rm u,P}}$$

when it exists.

[11] The conversion factor is 1.537, using https://www.x-rates.com for the exchange rate between 2013 GBP and 2013 USD (averaged over the year), and Johnston and Williamson (2019) for deflating 2013 USD to 2012 USD.

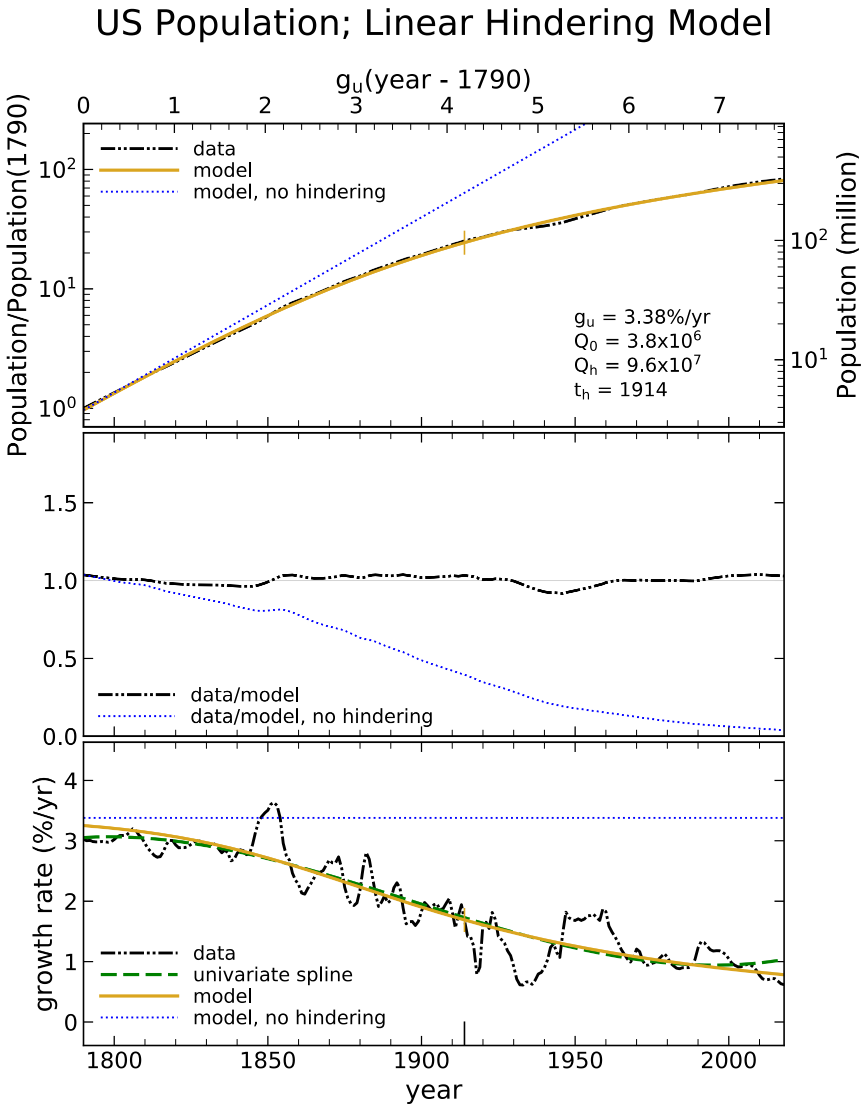

Fig. 2. US Population, 1790–2018. *Top*: Population data shown in dashed-dotted-dotted line. The y-axis on the left shows the population normalized to its initial value, the one on the right the actual population. The best-fitting model (solid line) is linear hindering, its relevant properties are listed at the lower right corner; the coefficient of determination is $R^2 = 0.9978$. A vertical marker shows the location of $Q = Q_h$, the corresponding year (1913) is marked on the bottom panel x-axis. Removing the hindering effect yields the dotted line, labeled "model, no hindering". *Middle*: Ratio of the data to the models displayed in the top panel, as labeled. *Bottom*: The growth rates of the data and each of the models displayed in the top panel, as labeled. A smoothing-spline fit to the data is shown in dashed line.

$7.4{\cdot}10^{-42}$, providing overwhelming evidence for a monotonic downward trend in the growth rate. The UK population growth is strongly affected by hindering.

The UK population also stands out as the only quantity best fitted with logistic hindering, with a carrying capacity of $Q_C = 6.9 \cdot 10^7$. The model result for 2018 is $Q_N = 6.24{\cdot}10^7$, which is 90.4% of the carrying capacity so that the largest value of the hindering factor (Eq. 25) is $f(Q_N) = 9.44$. As is evident from Fig. 5, the logistic model with the listed parameters provides an excellent fit to the data: the coefficient of determination is $R^2 = 0.9966$, the average deviation of model from data is only 1.59%, and for the most part the model-derived growth rate is almost identical to the spline smoothing of the data. In each panel, the significance of the hindering effect is evident from the model with the hindering effect removed. Without the logistic hindering correction, the 'de-hindered' model over-predicts the 2018 population by more than a factor of 8. For the year 2050 the logistic model predicts a population of 65 million growing at 0.10% per year. Truncating the time series all the way to 1937, the 2050 predictions stay within 10% of those for the full series.

Yet in spite of its overall success, the logistic model shows some systematic deviations from the data during the late stages of the time series. The model cannot reproduce the upward trend in the growth rate evident since around 1980. And beginning in 2006, the population has been exceeding the model prediction the by an amount that is systematically

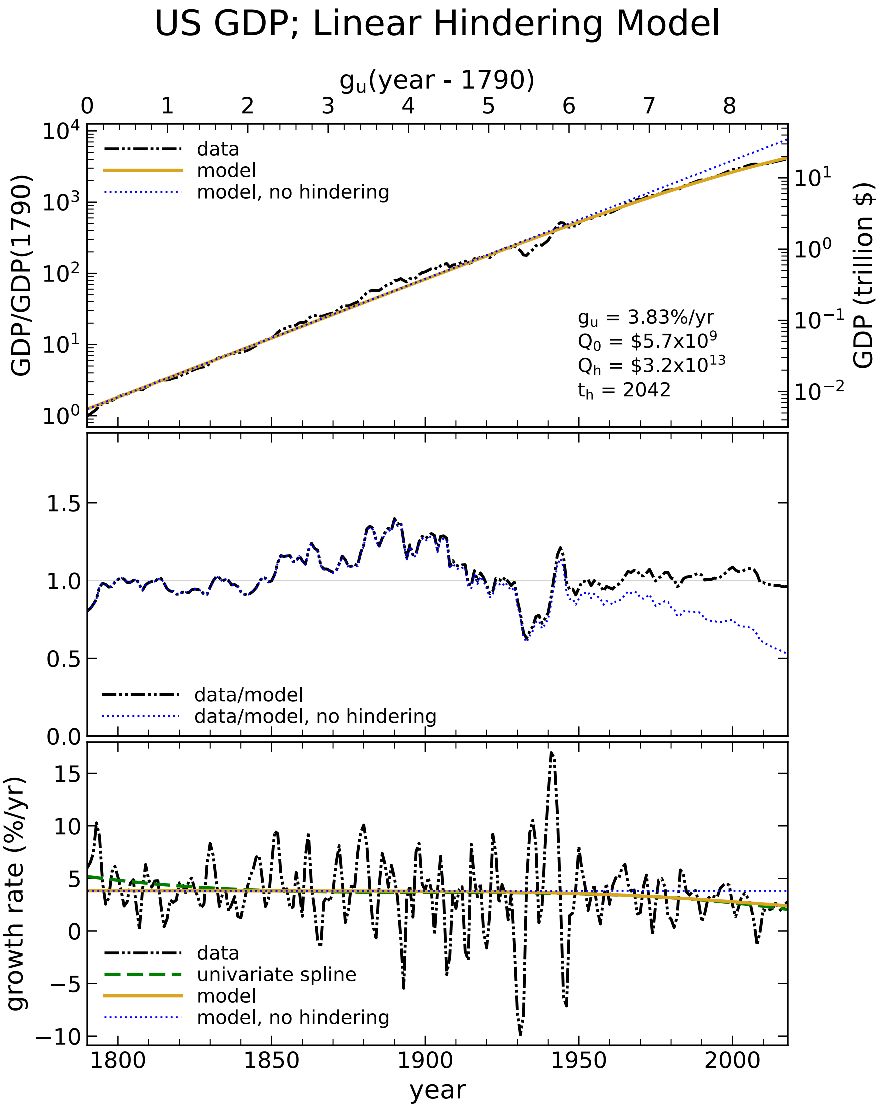

Fig. 3. US GDP, in 2012 dollars, from 1790–2018. The best-fitting model is linear hindering, yielding $R^2 = 0.9969$. Notations in this and the next four figures are as in Fig. 2.

increasing every year. The 2018 actual UK population of 66.5 million not only is higher than that year's model result of 62.4 million, it is even higher than the model prediction for the year 2050, which is only 65 million, and not much lower than the model absolute upper limit of $Q_C = 69$ million people. A few more years of data would be necessary to determine whether the rather successful logistic model of the currently available UK population data will require modifications.

3.2.2. UK GDP

The UK GDP has grown from \$39.5 billion in 1801 to \$3.13 trillion in 2018 (all \$ signs are in 2012 dollars), an increase by a factor of 79. It is the only quantity studied here whose growth has not suffered hindering. The MK-test yields $Z = 1.28$, rejecting the hypothesis of a monotonic downward trend in the growth rate with a p-value of 0.90. The test shows decisively that thus far hindering has not played a significant role in the UK GDP growth. This is evident also from the bottom panel of Fig. 6, where the growth rate data fluctuate around their flat smoothing spline. As an additional check we fitted the data with the linear hindering model. The resuling hindering parameter $Q_h = \$1.28 \cdot 10^{19}$ (predicted to be reached only in the year 2836!) yields $f(Q_N) = 2.18 \cdot 10^{-7}$, a vanishingly small hindering factor. Since the hindering correction is entirely negligible, the data should be modeled with a constant growth rate and a purely exponentially rising GDP. Indeed, as is evident from Fig. 6, such a model provides an adequate fit to the data: The top panel shows that it properly describes the long-term growth of the GDP; the average deviation of model from data, shown in the mid-panel, is 9.15%; the constant growth rate,

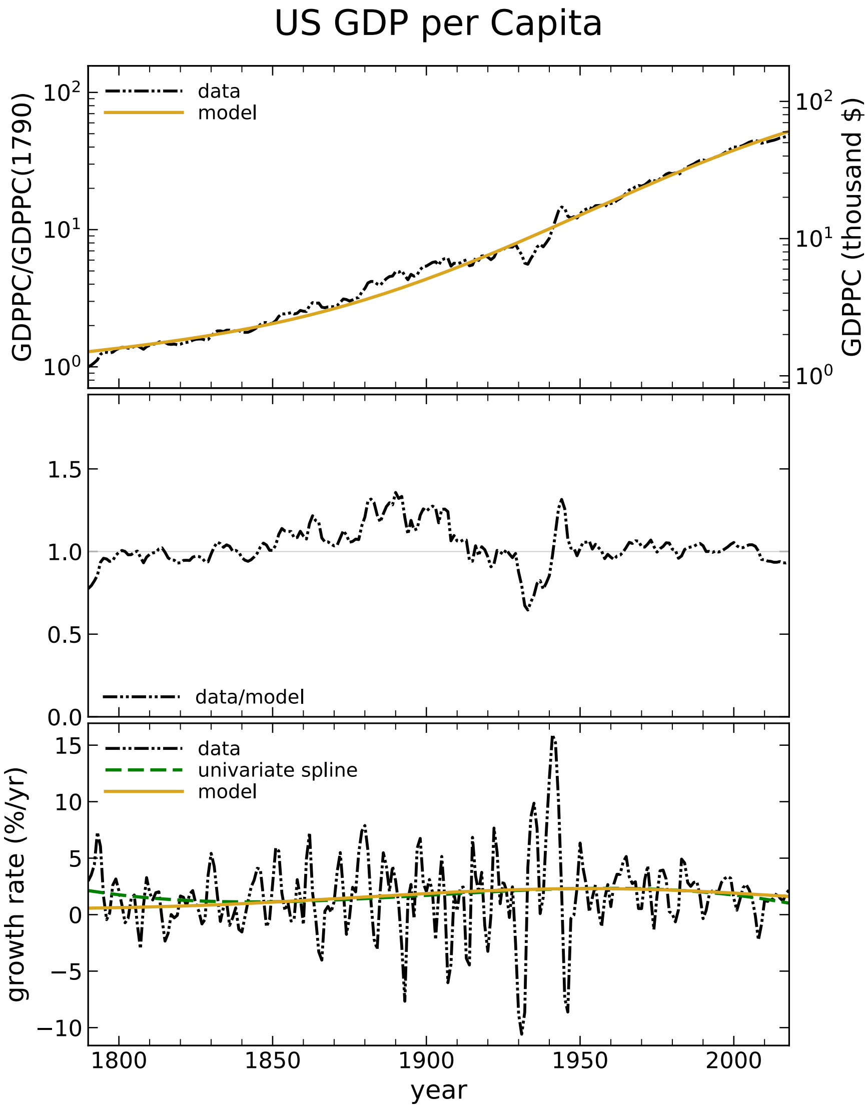

Fig. 4. US GDP per capita, in 2012 dollars, from 1790–2018. The model is the ratio of the best-fitting models for US GDP (Fig. 3) and population (Fig. 2). It has $R^2 = 0.9938$.

2% per year, is barely distinguishable from the smoothing spline in the bottom panel; the coefficient of determination is $R^2 = 0.9681$.

In contrast with the US, the growth of UK GDP shows no signs of hindering. It may be noteworthy that the time series of the UK GDP spans just over 4 unhindered growth times (Eq. 12; see top axis of Fig. 6) while the corresponding span for the US GDP is almost 9 growth times (see Fig. 3) because of its higher unhindered growth rate. Although the UK GDP could possibly display hindering effects in the future, the current data do not yet provide any significant indication for that. With a constant growth rate of 2.00% per year into the foreseeable future, the model predicts a UK GDP of $5.19 trillion in 2050. This prediction varies by less than 10% with the data truncated all the way back to 1993.

3.2.3. UK GDP per capita

The UK GDP per capita has grown from $3,378 in 1801 to $47,031 in 2018 (2012 dollars), an increase by factor of 13.9. Because of the relatively small change in population, the GDPPC variation is dominated by the increase of GDP. Conversely, if GDP is viewed as the product of population and GDPPC, since the population increased by only a factor of 5.7, the increase of GDP was dominated by the rise in GDPPC and not population, the opposite of the case in the US.

Fig. 7 shows the data with our model, obtained from the ratio of the two models for the UK GDP and population. The model captures reasonably well the long term variation of GDPPC: the coefficient of determination is $R^2 = 0.9621$, the average deviation of model from data is 9.23% and the bottom panel shows that the model growth rate can be barely

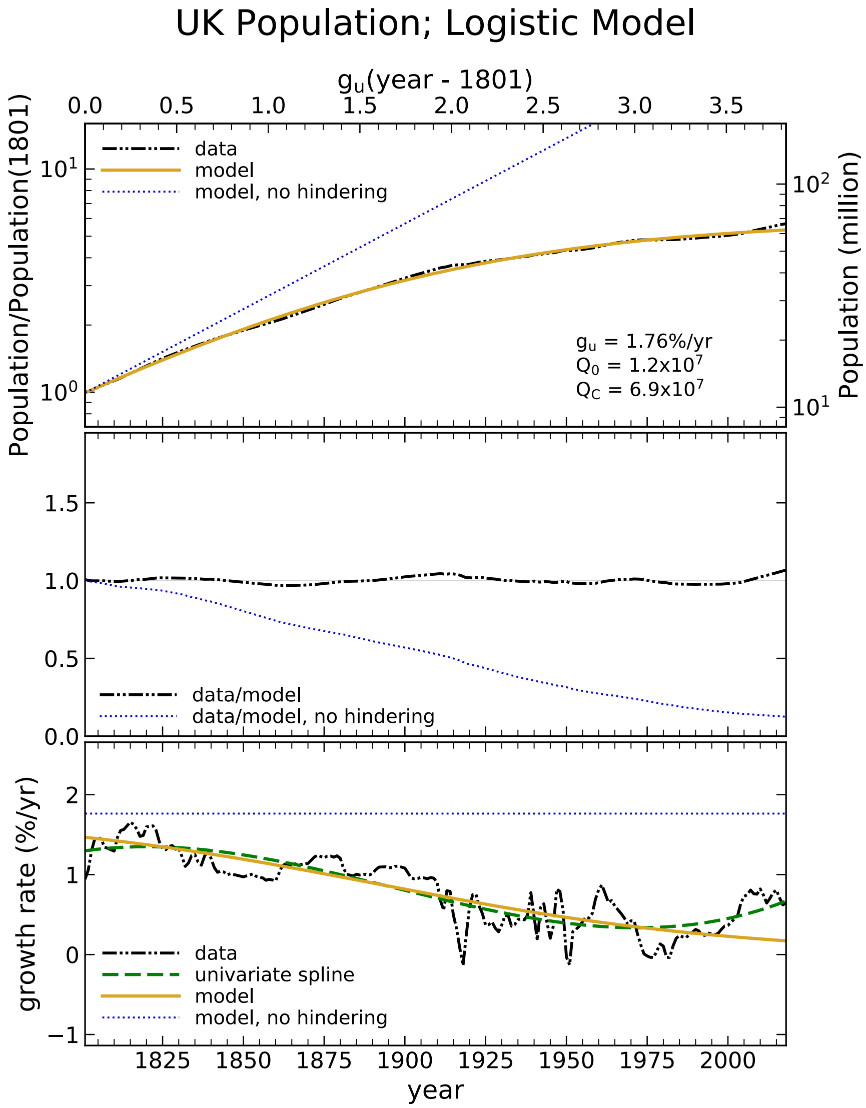

Fig. 5. Variation of the UK population from 1801–2018. The best fitting model is the logistic function with the listed properties; it has $R^2 = 0.9966$.

distinguished from the spline smoothing of the data. Since the UK GDP increases exponentially at a constant growth rate while the population growth rate is decreasing, the growth rate of the UK GDPPC is itself continually increasing. Our model predictions for 2050 are a UK GDP per capita of $79,852, growing at a rate of 1.90% per year.

4. Summary and discussion

We have derived here the general solution of the growth equation in parametric form (Eq. 21). The solution extracts the exponential behavior out of any growth pattern, enabling a systematic, methodical treatment of the remainder, described by the hindering factor f (Eq. 4). Unbounded growth is produced by hindering factors with finite series expansions (Eq. 5); its simplest form is linear hindering (Section 2.1), characterized by growth toward a stage in which the system reaches maximum efficiency in utilizing its resources (see Eq. 13 and subsequent discussion). Growth bounded by an upper limit is characterized by a hindering factor with an infinite series expansion; a prime example is the widely used logistic (Section 2.4).

With this solution of the growth equation we devised a data analysis algorithm to find the minimal number of parameters for modeling a given dataset. The algorithm is broadly divided into three stages (Section 3): test the data for compliance with the assumptions underpinning our solution; fit the data with the solution; assess the goodness of the fit. We applied our method to four case studies: population and GDP data of both the US and UK. Table 1 summarizes the key results. The columns under the common heading “MK-test of growth rate” summarize a preliminary data test, conducted before

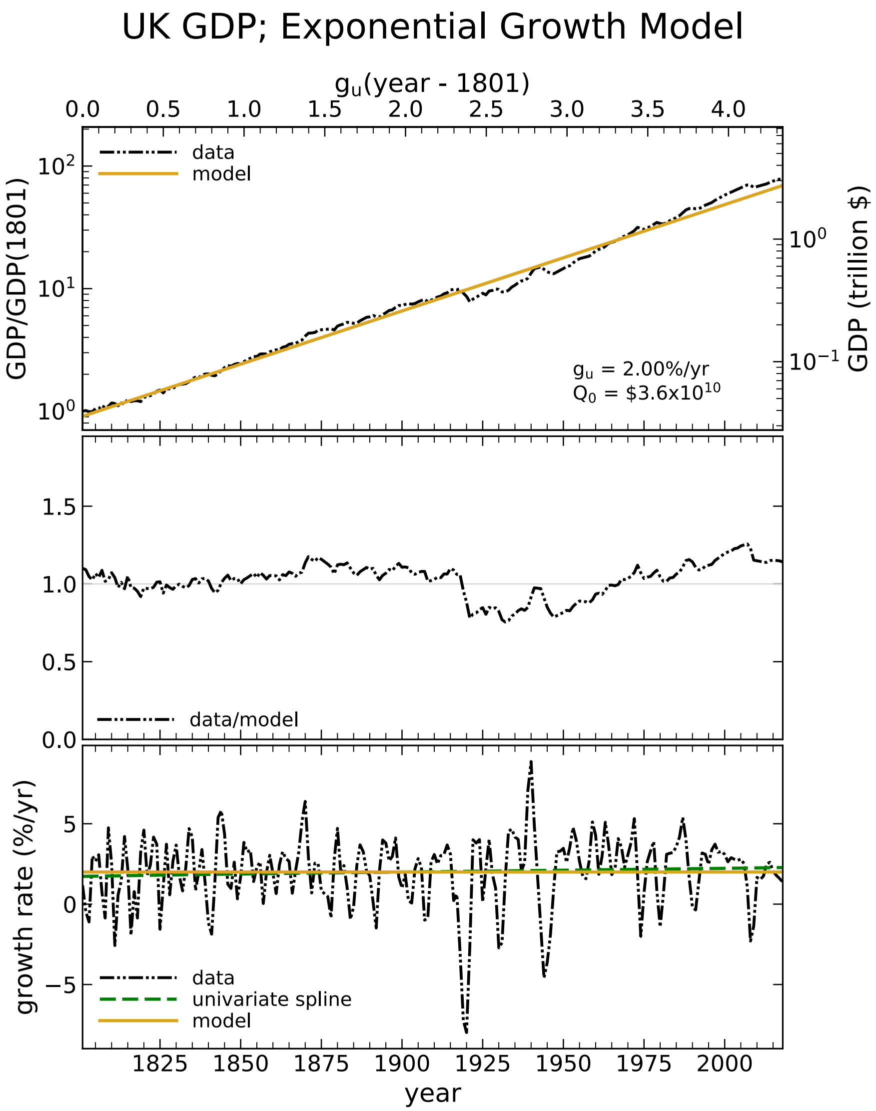

Fig. 6. UK GDP, in 2012 dollars, from 1801–2018. The best-fitting model is a pure exponential with a constant growth rate of 2% per year. It has $R^2 = 0.9681$.

Table 1
DATA ANALYSIS SUMMARY.

	MK-test of growth rate[a]			hindering factor[b]			
	Z	p-value	hindering	Q_N/Q_h[c]	$\alpha_2 Q_N/Q_h$	$f(Q_N)$	R^2[d]
UK GDP	1.3	0.90	no	$2.2{\cdot}10^{-7}$	——	$2.2{\cdot}10^{-7}$	0.9681
US GDP	-3.4	$3.79{\cdot}10^{-4}$	yes	0.61	$4.39{\cdot}10^{-15}$	0.61	0.9969
US population	-16.5	$1.28{\cdot}10^{-61}$	yes	3.34	$3.12{\cdot}10^{-23}$	3.34	0.9978
UK population	-13.5	$7.40{\cdot}10^{-42}$	yes	0.90	——	9.44	0.9966

Notes: Summary of data analysis results; for details see Section 3.1 for the US entries, Section 3.2 for the UK

[a] For a description of the MK-test see Section 3

[b] The hindering factor f (Eq. 3) of the largest value of each time series, Q_N. Individual terms correspond to the quadratic-hindering form (Eq. 28) for the top three entries (with $\alpha_2 = 0$ for the UK GDP), and the logistic one (Eq. 25) for the UK population.

[c] The entry for the UK population is Q_N/Q_C (see Eq. 22)

[d] Coefficient of determination; see Eq. 27

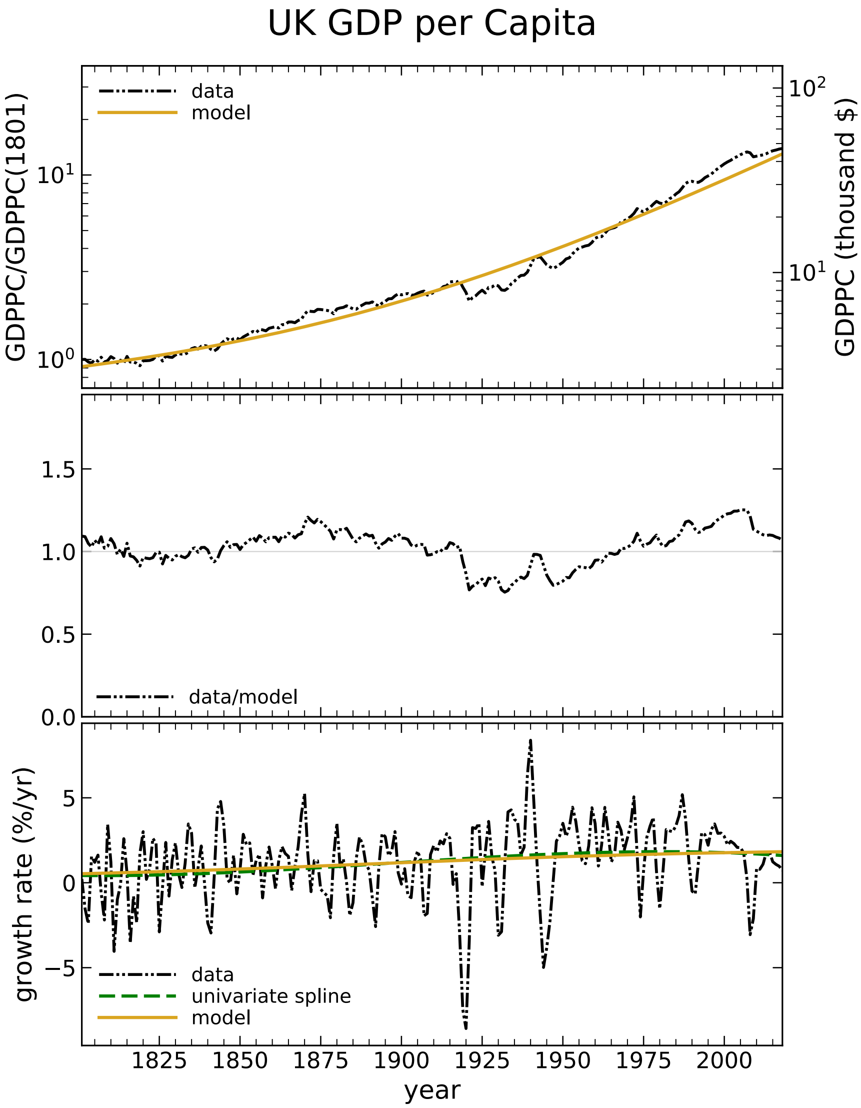

Fig. 7. UK GDP per capita, in 2012 dollars, from 1801–2018. The model is the ratio of the best-fitting models for UK GDP (Fig. 6) and population (Fig. 5). It has $R^2 = 0.9621$.

the actual fitting. They show that hindering has affected all the studied quantities except for the UK GDP. Results of actual modeling are summarized under the heading "hindering factor". For each dataset, the series expansion of the hindering parameter is terminated when the ratio of successive terms obeys the "stopping rule" in Eq. 26. For the UK GDP, the vanishingly small value of the linear hindering term confirms the MK-test conclusion that hindering can be neglected. For the US, both GDP and population require a significant linear-hindering term, while quadratic hindering is entirely negligible. The UK population stands out as the only quantity whose best-fitting model is the logistic. In the final stage of the analysis we calculate the coefficient of determination to gauge the quality of the fitting. The table entries for R^2 show that in all cases the models account for the bulk of the variability of the dependent variable.

Although the MK-test is not a part of the fitting, it provides a valuable consistency check—the presence or absence of significant hindering in the model results should comport with the conclusions of the MK-test. This test establishes the presence of hindering at the 99% confidence level whenever its Z-statistic obeys $Z < -2.5$; the larger is $-Z$ the stronger the impact of hindering. Indeed, the tight inverse correlation between $f(Q_N)$, the largest value of the hindering factor of each model, and Z stands out in Table 1. The implication is that the Z-statistic, derived directly from the data without any modeling, is a reliable quantitative measure of the degree of growth hindering. Dynamic performance of different economies is frequently compared by their current rates of growth, at least in the short run. The correlation uncovered here suggests that, alongside the growth rate g, Z could be adopted as an additional useful indicator of growth. *Just as g is a direct indicator*

Table 2
Model predictions for the year 2050.

	GDP[a]	Population	GDP per capita[a]
US	$37.4 trillion; 1.77% (h)[b]	400 million; 0.65% (h)	$93,520; 1.12%
UK	$5.2 trillion; 2.00% (e)	65.0 million; 0.10% (l)	$79,852; 1.90%

Notes: First entry in each case is the quantity, second is its annual growth rate. The letter in parenthesis indicates the best-fit model used for the prediction: e–pure exponential; h–linear hindering; l–logistic

[a] Currency units: $ = 2012 USD

[b] The US GDP dataset is described equally well by a logistic model (see Section 3.1.2), whose predictions for 2050 are GDP of $31 trillion and growth rate of 1.03%

of the instantaneous state of the growing quantity, Z is a direct, model-independent indicator of the overall long-term trajectory of its growth.

Our results show that adding just a single parameter can characterize long-term growth phenomena for both the US and the UK, the countries with the longest continuous data sets. Surprisingly, the rate of GDP growth in the UK has stayed constant for more than 200 years (Section 3.2.2). However, the US GDP as well as the populations of both the US and the UK display declining growth rates, a manifestation of the effect we termed hindering. We find satisfactory descriptions for these declines with two single-parameter hindering functions: linear and logistic hindering. The US population provides the cleanest example of linear hindering—logistic hindering is decisively eliminated in this case because its upper limit has long been surpassed (Section 3.1.1). The UK population growth is best described by logistic hindering (Section 3.2.1), while the case of US GDP is currently undecided since both hindering functions provide equally successful descriptions; settling this issue will require a few more years of data (Section 3.1.2). Looking at over 200 years of GDP and population data for these two nations, we also find that adding more parameters has a negligible impact.

Table 2 summarizes the 2050 predictions for the US and UK GDP and population from our best-fitting models; as shown in each case (Sections 3.1, 3.2), these predictions do not vary appreciably even when significant portions of the time series are discarded. In the case of US GDP, based on current data two models are equally likely and the tabulated results are from the linear hindering model, while the logistic model predictions are listed in the table notes. The two values differ by 17% and can be considered the upper and lower bounds of our model predictions for the US GDP in 2050. For comparison, the report *The World in 2050* by Price Waterhouse Cooper[12] forecasts for the US a GDP of $32.2 trillion and a population of 389 million, for the UK the respective values are $5.1 trillion and 75.4 million.

4.1. Advantages and limitations

The framework we developed here is premised on a single assumption: The quantity Q is monotonically increasing ($dQ/dt > 0$, $g > 0$). This assumption implies that Q is a single-valued function of time, therefore g can be taken as a function of Q instead of t. This substitution effects a separation of variables for the growth equation (Eq. 1), evident in its transformed form in Eq. 4: The function of Q on the left-hand-side varies linearly with time. Its logarithmic component describes exponential growth with the constant growth rate g_u, while the term involving the hindering factor f describes the deviations from such a pure exponential—the hindering factor contains all the information about the deviations of the growth pattern from pure exponential. Since this factor is a simple dimensionless re-write of the growth rate (Eq. 3), it can be expanded in a Taylor series under a broad set of circumstances, leading to the general solution of the growth equation in parametric form (Eq. 21). *Any arbitrary growth pattern can be described with this parametrization so long as Q remains monotonically increasing.* This solution provides a generic description of growing quantities just as the Fourier series provides a generic description of periodic phenomena. As a general mathematical solution it should be applicable in a wide variety of growth situations, including, for example, biological and physical systems. Indeed, the impetus for our work came from the growth of laser and maser radiation[13], where growth equations are derived from first principles of radiation theory that describes the dynamics of the underlying physical processes (Elitzur, 1992). In that case the solution is linearly-hindered growth (Section 2.1), with the parameters g_u and Q_h derived from coefficients that describe various aspects of fundamental interactions between matter and radiation.

Just as Fourier analysis provides a useful description of periodic phenomena but does not address the reasons behind the periodicity, our hindering formalism provides a description, not a theoretical explanation, of growth. Also, our approach deals exclusively with long-term trends, ignoring the fluctuations about trend lines. The strength of our formalism is not in reproducing all the details in the data, but in constructing analytic description of long-term growth with the minimal number of free parameters. Any set of N data points can be reproduced with a polynomial of order $N-1$ whatever the data

[12] https://www.pwc.com/gx/en/issues/economy/the-world-in-2050.html#data. Their GDP values are given in terms of 2016 dollars, which we convert to 2012 dollars with a deflator of 0.9440 from Johnston and Williamson (2019).

[13] The word laser is acronym for Light Amplification by Stimulated Emission of Radiation. Similarly, masers involve Microwave instead of Light. Requiring special conditions on earth, maser amplification occurs naturally in many astronomical sources; a popular exposition is available in Elitzur (1995).

origin, therefore such analytic representation conveys no information about the underlying phenomenon. In contrast, fitting with the hindering formalism is narrowly targeted, since it is based on the solution of the growth equation, and brings out the inherent properties of the underlying growth process. While adding another point to the dataset would require an entirely new, higher order polynomial, a hindering fit would hardly change. Indeed, the research reported here spanned a number of years with every year adding another point to each dataset, yet every parameter of every model remained unchanged to within a fraction of a percent. Like Fourier analysis and periodic phenomena, the hindering formalism distills the essence of growth processes.

Our approach is built on the description of growth with a continuous mathematical function. The data points are assumed to be sampling this function, and a meaningful determination of its parameters requires a proper sampling (equal time spacing is not a requirement). For example, proper determination of the parameters that produced the linear hindering plots in Fig. 1 requires sets of points that straddle the $Q = Q_h$ markers: the hindering parameter Q_h cannot be determined if all points are $\ll Q_h$; and when they all are $\gg Q_h$, the product $g_u Q_h$ is determinable but neither is g_u nor Q_h separately. Whether a given dataset does provide adequate sampling can only be determined a-posteriori, as illustrated by our results. For the US population, the initial 60% of the data (until 1923) would suffice to produce model results deviating less than 10% from those for the full dataset (Section 3.1.1). Thanks to the onset of hindering roughly in the middle of the series ($Q = Q_h$ was surpassed in 1913), both unhindered and hindered growth phases are well sampled in this case and the functional form of the hindering factor is well determined. In contrast, the hindering of US GDP growth is still at early stages (Section 3.1.2) and the effects of hindering are barely sampled. As a result, duplicating our model results to within 10% requires data until at least 1993, and the exact form of the hindering factor is still not fully determined—linear and logistic hindering are equally likely, and it will take another ~ 10 years or so of data to enable a meaningful choice between them. Considerations of this kind can be used as indications of the robustness of model results.

Our formalism complements structural models such as the ones described in Rossi-Hansberg (2019). These models integrate micro-behavior of firms and individuals over space and time, taking into account processes of technological change and agglomeration results to obtain the rate of growth of GDP and population. They identify conditions that lead to a constant rate of growth in the long run and decompose this rate of growth into different components. One measure of the effectiveness of these models is the extent to which they can derive dynamic growth parameters that are consistent with the aggregate growth parameters that we obtain using our formalism. It must be noted, though, that most of the cases studied here give decisive evidence for decreasing, rather than constant, growth rates (see, e.g., Table 1). Therefore detailed models would have to identify and incorporate processes that lead to declining growth rates to produce realistic models in those cases too. For example, a successful structural model of the US population growth will have to accord with our descriptive model results because both would describe properly the same dataset. Evaluating the hindering factor (Eq. 3) of such a model would produce g_u and Q_h in terms of US birth rates, life longevity etc., providing insight into the fundamental reasons why the US population growth follows a simple linear hindering pattern with a hindering parameter of 96 million people.

The modeling results presented here uncover fundamental trends with profound implications. In spite of their highly volatile growth rates, the time variations of both GDP and population in both the US and UK are rather smooth and regular. This in itself does not guarantee simple analytic descriptions—our best fitting models could have ended up involving large numbers of free parameters. Yet in each and every case studied here, the deviations of the growth pattern from a pure exponential do not require more than a single free parameter. The persistence of such simple long-term growth patterns in both the US and the UK stands out and is consistent with the findings by Jones (1995), who suggested that macroeconomic policies did not have persistent impacts on growth[14] Jones arrived at his conclusion based on the observation that the growth rate of US GDP per capita was essentially constant from 1880–1987. The simple expression we derive for this growth rate in Section 3.1.3 (a difference of two linearly-hindered terms) describes it adequately over the longer period 1790–2018, and is indeed roughly constant during the period studied by Jones (see Fig. 4). Our results generalize the Jones conclusion, showing that it does not require constant growth rates and that it applies separately to both GDP and population, not just their ratio.

Our findings raise some fundamental questions for theorists. The striking simplicity of the growth patterns has persisted in spite of massive upheavals during the covered periods,[15] which include the spread of industrialization, waves of immigration and two world wars. The British empire reached both its apogee and ultimate demise, yet neither has left an indelible mark on the growth of either UK population (Fig. 5) or GDP (Fig. 6). While the trauma of the Great Depression is clearly discernible in the GDP plots of both the US (Fig. 3, top panel) and the UK (Fig. 6), in each case the GDP time variation ends up reverting to the same simple function (different for the two nations) that described earlier epochs. How to explain such persistent patterns of inherently smooth growth among different nations and quantities? How to explain the differences in rates of growth, and decompose them as a function of time? What factors cause the rate of growth to remain constant or decline approximately at a constant rate despite changing events? Why is the logistic model so good for the UK population to 1980 (the average deviation of model from data is only 1.4 percents!) and what explains the rise in its growth rate around

[14] One approach to explaining smooth GDP growth in the longer-run is to combine Acemoglu et al. (2005), which suggested that economic institutions set patterns for long-term growth, with traditional growth theory originated by Solow (1956). This is beyond the scope of this paper.

[15] Interestingly, for the US GDP, the average deviation of data from our model is 9.71% for the 1790–1900 period, 14.98% from 1901–1949 but only 3.49% during 1950–2018.

that time? While our formalism cannot address these issues, it does uncover and highlight these fundamental challenges to detailed theory.

With simple descriptions that rely on a few key parameters, the results of our analysis can be used broadly to inform policy makers and the general public of long-term phenomena for comparative purposes. We observe that in some countries we may see a constant rate of growth, while in others there may be hindering or even an absolute upper bound (logistic growth). Our methodology should be applied to lengthy time series data of different variables and for different locations. Here we concentrated on the US and UK, the only nations for which there are reliable annual GDP and population data spanning more than 200 years. Time series approaching 200 years exist for some other nations, and preliminary analysis indicates that our hindering formalism works successfully for many of those too. We plan to complete this analysis and present its results in a future publication.

Acknowledgments

We thank Željko Ivezić, Ronald Lee, Ethan Ligon, Alan Olmstead, Dejan Vinković and Yang Xie for providing most useful comments on the manuscript.

Appendix A. Hindering functions

Just as the exponential function can be defined from $y' = y$ and the trigonometric function from $y'' = -y$, introduce the *linear-hindering function* h_1 defined from

$$\frac{dh_1}{dx} = \frac{h_1}{1+h_1}, \qquad \text{with} \quad h_1(0) = 1. \tag{A.1}$$

The solution of this equation provides an analytic, albeit implicit, expression for h_1:

$$\ln h_1 + h_1 - 1 = x. \tag{A.2}$$

When $h_1 \to 0$, $h_1' \to 0$ and $x \to -\infty$ so h_1 is never negative and h_1' is always > 0; that is, h_1 is a monotonically increasing function, rising from 0 at $x \to -\infty$ to 1 at $x = 0$, and on to ∞ when $x \to \infty$. It is a single-valued function that covers all positive numbers and can be calculated numerically from Eq. A.2 using an iterative procedure. The auxiliary function

$$\tilde{h} = \begin{cases} e^x & x \le 0, \\ x + 1 - \ln(x+1) & x > 0 \end{cases} \tag{A.3}$$

provides a useful start for such a calculation: with $\tilde{h}$ as an initial guess, an iteration scheme such as the Newton method is both simple and effective. The function h_1 is plotted in the top panel of Figure A8. Its inverse function is straightforward—given an arbitrary $H > 0$, the value of x that produces $h_1(x) = H$ is simply

$$x = \ln H + H - 1, \tag{A.4}$$

as is evident from comparison with Eq. A.2.

Given g_u and Q_h and introducing $t = x/g_u$ and $Q = Q_h h_1$, equation A.1 can be recognized as the growth equation (Eq. 1) with the linear hindering growth rate from Eq. 7. Thus the linear hindering solutions introduced in Section 2.1 (Eqs. 8 and 11) can be written explicitly in terms of the mathematical function h_1, as done in eqs. 8′ and 11′. These explicit expressions show that, given the parameter-free mathematical function $h_1(x)$, linear-hindering growth of any arbitrary system is fully prescribed by the three parameter g_u, Q_h and t_h in complete analogy with the common example of a wave: the universal trigonometric function is transformed into any physical wave by scaling the axes with the amplitude and frequency, analogous to Q_h and g_u, and shifting the time origin by the phase, similar to t_h. In both cases, the two axis-scaling parameters (g_u and Q_h in the present case) are intrinsic to the differential equation that defines the mathematical function, the third parameter (t_h for the linear hindering) is determined by the boundary conditions. Likewise, every plot presented in Fig. 1 can be obtained from the plot of h_1 in the top panel of Fig. A.8 by scaling the y-axis by the corresponding Q_h/Q_0 and shifting the origin by the appropriate amount.

The linear hindering function h_1 is generated by the linear hindering factor (Eq. 6). A hindering factor that is a power-law of order k will generate the k-th order hindering function h_k, defined from

$$\frac{dh_k}{dx} = \frac{h_k}{1+h_k^k}, \qquad h_k(0) = 1. \tag{A.5}$$

That is, the k-th order hindering function is given by

$$\ln h_k + \frac{1}{k}\left(h_k^k - 1\right) = x. \tag{A.6}$$

Irrespective of the order k, h_k is a monotonically increasing function, rising from 0 at $x \to -\infty$ to 1 at $x = 0$, and on to ∞ when $x \to \infty$; it is a single-valued function that covers all positive numbers. The behavior of h_k when $x \to -\infty$ is $h_k \simeq$

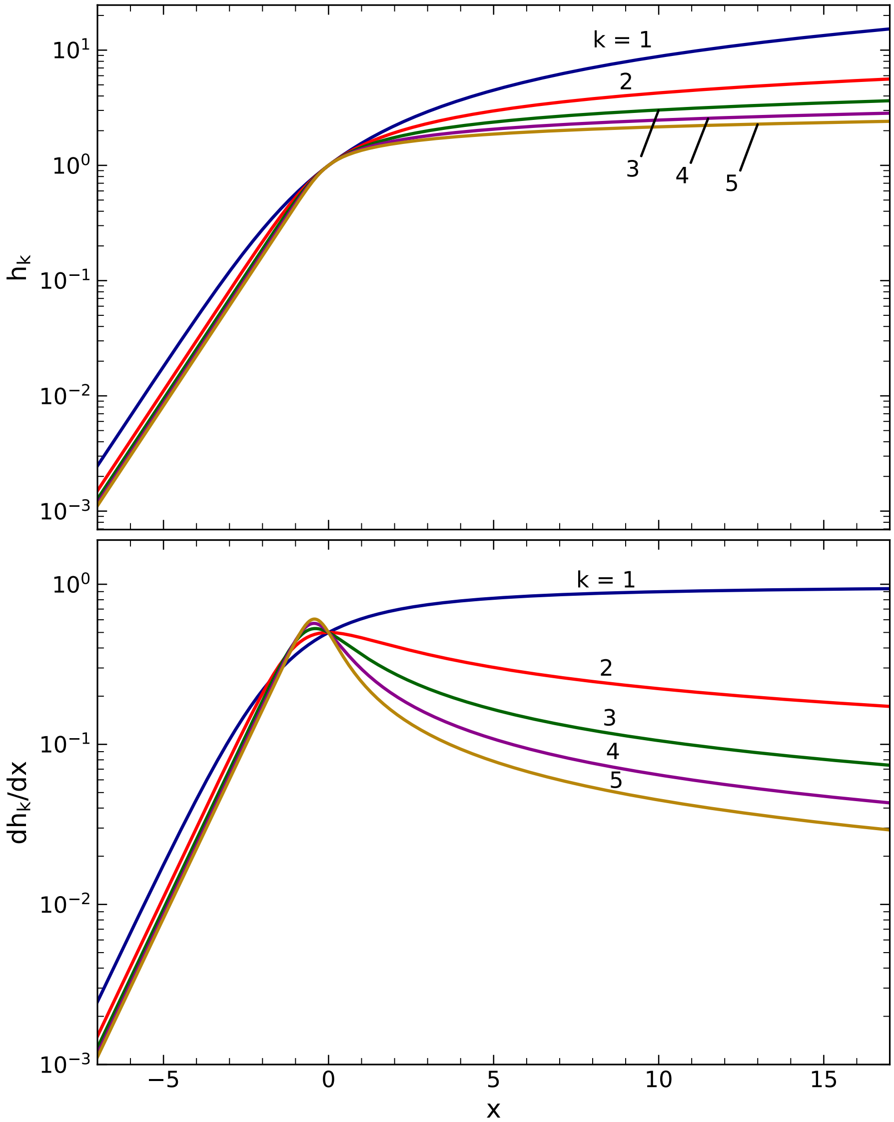

Fig. A8. Hindering functions h_k (Eq. A.6) for $k \le 5$. *Top*: Plots of h_k, as labeled. *Bottom*: The corresponding derivatives h'_k (Eq. A.5). When $k > 1$ the derivative has a maximum whose properties are listed in Eq. A.7.

$e^{x+1/k}$ while the asymptotic behavior at $x \to \infty$ is $h_k \simeq (1 + kx)^{1/k}$; the larger is k, the flatter the rise. The top panel of Fig. A.8 shows plots of h_k for $k \le 5$, showing that the large-x behavior of h_k becomes flatter with increasing k.

Because h_k is monotonically increasing, Eq. A.5 shows that its derivative, too, is increasing when $h_k \ll 1$ ($x \ll 0$) for all k. But when $h_k \gg 1$, i.e., $x \to \infty$, the derivative obeys $h'_k \sim h_k^{-(k-1)}$ and thus is decreasing with x, except for the case $k = 1$ where it is constant. Linear hindering is the only case where the derivative approaches a constant. For all other k, h'_k is decreasing when $h_k \gg 1$, therefore it has a maximum around $x \sim 0$. Denote by x_k the location of this maximum, i.e., $h'_k(x_k) = h'_k\big|_{\max}$. The value of x_k is derived from the vanishing of the second derivative, yielding ($k > 1$)

$$x_k = -\frac{1}{k}\left(\frac{k-2}{k-1} + \ln(k-1)\right), \qquad h_k(x_x) = \frac{1}{(k-1)^{1/k}}, \qquad h'_k\big|_{\max} = \frac{1}{k}(k-1)^{\frac{k-1}{k}} \tag{A.7}$$

The bottom panel of Fig. A.8 shows the derivative h'_k for each of the displayed h_k. Note that $h'_k(0) = \frac{1}{2}$ for all k, a result that follows directly from the definition of h_k in Eq. A.5.

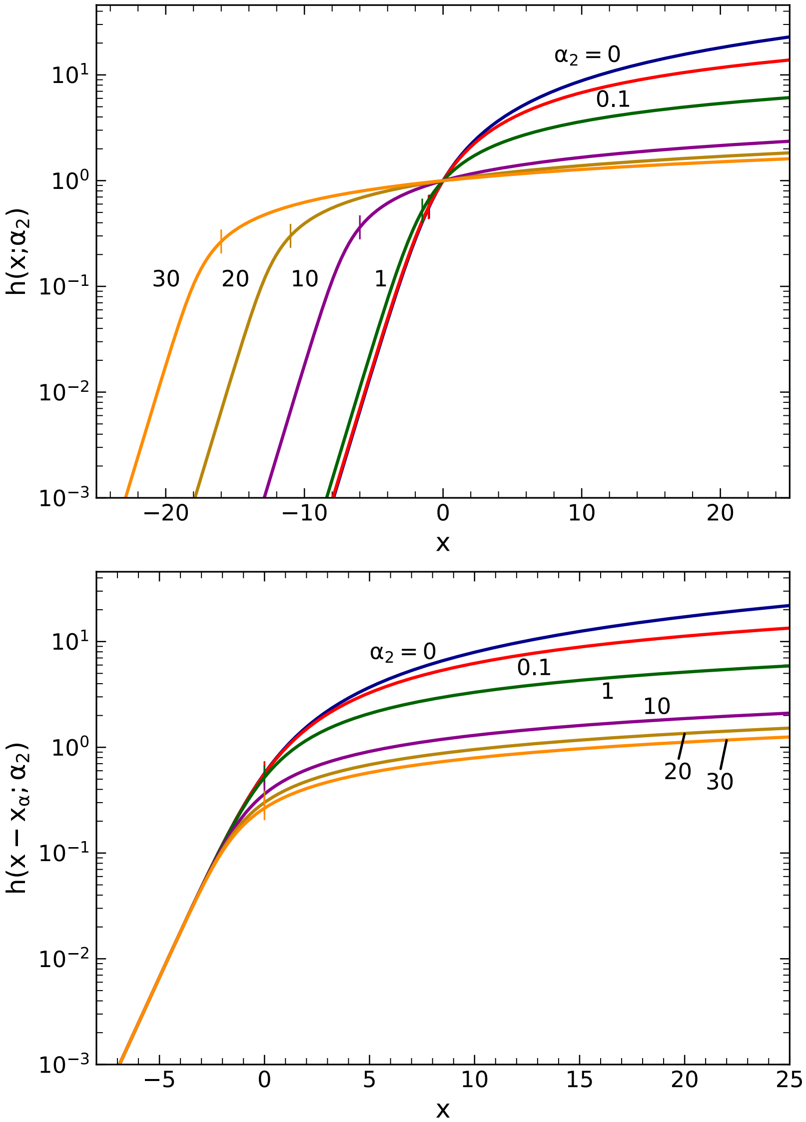

Fig. A9. The general quadratic-hindering function $h(x;\,\alpha_2)$ (Eq. A.10). *Top*: Plots of for various values of α_2, as labeled; note that $h(x;\,0) = h_1(x)$, the linear hindering function (Eq. A.2). Each curve contains a marker at $x = -x_\alpha$ (Eq. A.11). *Bottom*: Same as the top panel only the argument of each plot is shifted by x_α, bringing all the markers to alignment at $x = 0$.

The most general hindering factor involves a combination of power laws defined by the set of expansion coefficients α_k (Eq. 20). The resulting general *hindering function* h is defined from the differential equation

$$\frac{dh}{dx} = \frac{h}{1 + \sum_{k=1}^{\infty} \alpha_k h^k}, \qquad \text{with} \quad h(0) = 1; \tag{A.8}$$

here $\alpha_k \geq 0$ and $\alpha_1 = 1$. The solution provides an implicit analytic expression for h:

$$\ln h + \sum_{k=1}^{\infty} \frac{1}{k} \alpha_k (h^k - 1) = x. \tag{A.9}$$

As before, Eq. 21 is obtained from $Q = Q_\mathrm{h}\, h(g_\mathrm{u}[t - t_\mathrm{h}])$ (cf Eq. 11′) and, similar to Eq. 8′, the equivalent of Eq. 8 is obtained from $Q = Q_\mathrm{h}\, h\big(g_\mathrm{u}[t - t_0] + x_0\big)$ with a suitably defined x_0[16]

[16] The definition is now $x_0 = \ln \frac{Q_0}{Q_h} + \sum_{k=1}^{\infty} \frac{1}{k} \alpha_k \left[\left(\frac{Q_0}{Q_h} \right)^k - 1 \right]$. Similar to the inverse of h_1 (Eq. A.4), given an arbitrary $H > 0$, the solution of $h(x) = H$ is $x = \ln H + \sum_{k=1}^{\infty} \frac{1}{k} \alpha_k (H^k - 1)$, therefore $h(x_0) = Q_0/Q_\mathrm{h}$.

Linear hindering is the special case of h when $\alpha_k = 0$ for all $k > 1$, yielding $h = h_1$. The lowest-order multi-term combination for the hindering factor has $\alpha_2 > 0$, namely, a single-parameter mixture of quadratic and linear hindering. The result is the general quadratic hindering function $h(x;\ \alpha_2)$, given by

$$\ln h + h - 1 + \frac{1}{2}\alpha_2(h^2 - 1) = x; \tag{A.10}$$

obviously, $h(x; 0) = h_1(x)$. Fig. A.9 shows in its top panel plots of $h(x;\ \alpha_2)$ for various values of α_2, including the linear hindering case $\alpha_2 = 0$. When $0 < \alpha_2 \le 1$, the functions $h(x;\ \alpha_2)$ and h_1 are almost identical at $x < 0$ and deviate from each other only at $x > 0$, where $h(x;\ \alpha_2)$ has a flatter asymptotic behavior. When $\alpha_2 \gg 1$, the quadratic term dominates, and $h(x;\ \alpha_2)$ approaches the function $h_2(x)$ at $x > 0$. Each curve has a marker at $x = -x_\alpha$, where

$$x_\alpha = 1 + \frac{1}{2}\alpha_2; \tag{A.11}$$

this is the point where the magnitude of $\ln h$ on the left-hand-side of Eq. A.10 is equal to the combination $h + \frac{1}{2}\alpha_2 h^2$. In the bottom panel of Fig. A.9, the argument of each plot is shifted by x_α, bringing all markers to alignment at $x = 0$. To the left of this point the logarithmic term dominates, yielding the asymptotic behavior $h(x; \alpha_2) \simeq \exp(x + x_\alpha)$ for $x \ll -x_\alpha$. To the right of each marker, the algebraic terms dominate. In the case of $\alpha_2 = 0$ (linear hindering), the asymptotic behavior for $x \gg 0$, neglecting logarithmic corrections, is $h(x; 0) \simeq 1 + x$. When $\alpha_2 > 0$, the asymptotic behavior for $\alpha_2(x + x_\alpha) \gg 1$ is $h(x; \alpha_2) \simeq \sqrt{1 + (1 + x)/\alpha_2}$. Note that $h(x - x_\alpha; \alpha_2) \simeq h_2(x)$ when $\alpha_2 \gg 1$.

Appendix B. F-test analysis

Selecting the optimal model for a given dataset can be handled as a sequential parameter fitting problem. Truncating the series expansion of the hindering factor f (eq 5) after some term, say a_m, yields a model M_1 with p_1 free parameters. Truncating the series after the next term, a_{m+1}, produces a model M_2 with $p_2 = p_1 + 1$ free parameters. Since the addition of a free parameter can be expected in itself to improve fitting, we must determine the significance of the improvement by M_2 over its nested model M_1. This problem can be handled with the F-test (Maddala and Lahiri, 2009), which assesses the validity of the null hypothesis $a_{m+1} = 0$. From the residuals of the two models and the two degrees of freedom $d_1 = p_2 - p_1 = 1$ and $d_2 = n - p_2 - 1$, where n is the number of data points, we construct the F-statistic and compare it with the critical value $F_{\rm crit}$, a quantity determined by d_1, d_2 and the error level α. When $F > F_{\rm crit}$, the null hypothesis can be rejected at the confidence level $1 - \alpha$, the probability of a false rejection is less than α. The optimal number of series coefficients is achieved when the null hypothesis cannot be rejected, i.e., the improvement from adding another term is statistically insignificant.

In our case, the search for the optimal solution starts with the simplest model—a constant growth rate. In the first step we fit the data with an exponential and the two free parameters $g_{\rm u}$, the constant growth rate, which determines the time variation of $Q(t)$, and $Q_0 = Q(t_0)$, which sets the scale of Q. Next we move to the linear hindering model, which requires the additional free parameter a_1 (eq 5). With the aid of Eq. 8 we determine the best-fitting triplet of parameters $g_{\rm u}$, Q_0 and a_1 $(= 1/Q_{\rm h})$, and F-test the null hypothesis $a_1 = 0$; the corresponding numbers of free-parameters are $p_1 = 2$ and $p_2 = 3$. When the null hypothesis can be rejected, add the next term in the series expansion (quadratic hindering), F-test the null hypothesis $a_2 = 0$ with $p_1 = 3$ and $p_2 = 4$, and so on. We found it unnecessary to move beyond quadratic hindering in any of the cases considered here.

With $n = 229$ data points the US data set has $d_2 = 228 - p_2$. Given d_2 we calculate the F-test critical value $F_{\rm crit}$ as a function of the error level α. $F_{\rm crit}(\alpha)$ increases as α decreases, and the smallest error level the numerics could handle was $\alpha = 10^{-16}$, yielding the largest meaningful critical value $F_{\rm crit}(10^{-16}) = 80.63$ for the exponential vs linear-hindering comparison ($p_1 = 2,\ p_2 = 3$) and 80.69 for the linear- vs quadratic-hindering comparison ($p_1 = 3,\ p_2 = 4$). This implies that the null hypothesis can be rejected at the 10^{-16} error level whenever the two tested models produce a larger F-statistic. For the UK data $n = 218$, and the respective values of $F_{\rm crit}(10^{-16})$ are 81.31 and 81.38. We now describe the application of this method to each of the time series studied here.

- **US population:** Running the F-test on the best-fitting exponential and linear hindering models produces $F = 1.52{\cdot}10^4$, orders of magnitude larger than $F_{\rm crit}(10^{-16})$; taking all distributions in log gives an even larger $F = 3.81{\cdot}10^4$. Thus the null hypothesis $a_1 = 0$ is rejected overwhelmingly—the p-value of the F-test is much smaller than 10^{-16}, the rejection reliability is numerically indistinguishable from unity. The improvement achieved by adding the free parameter a_1 $(= 1/Q_{\rm h})$ is highly significant. In the next step, F-testing the linear- vs quadratic-hindering model yields $F = 1.50{\cdot}10^{-11}$, corresponding to a p-value of unity; the F-test is consistent with a fully verified null hypothesis $a_2 = 0$ within the numerical error. While linear hindering is essential for the modeling, quadratic hindering is entirely negligible, in complete agreement with the conclusions of Section 3.1.1.
- **US GDP:** F-testing the best-fitting pure exponential against the linear hindering model produces $F = 121.4$ so that the null hypothesis $a_1 = 0$ is rejected with a p-value smaller than 10^{-16}. Running the F-test on the best-fitting linear- and quadratic-hindering models yields $F = -6.46{\cdot}10^{-10}$, consistent with zero within the numerical accuracy and corresponding to $p = 1$. Thus the null hypothesis $a_2 = 0$ is fully verified. Similar to the US population, the best-fitting model must include linear hindering while quadratic hindering is irrelevant, in complete agreement with Section 3.1.2.

- **UK Population:** Running the F-test on the best-fitting exponential and linear-hindering models produces $F = 1996$, more than 20 times larger than $F_{\text{crit}}(10^{-16})$. The null hypothesis $a_1 = 0$ is rejected overwhelmingly with a p-value much smaller than 10^{-16}, showing that the effect of hindering must be included. In this case the single parameter logistic hindering factor (Eq. 25) yields a solution whose error estimate is 4 times smaller than the best-fitting linear-hindering model, therefore there is no point in exploring the significance of the quadratic-hindering term. The preferred model is the logistic, as found in Section 3.2.1.
- **UK GDP:** The F-test of the best-fitting exponential and linear-hindering models yields $F = -8.84{\cdot}10^{-6}$ and $p = 1$, consistent with a complete validation of the null hypothesis $a_1 = 0$ within the numerical accuracy. Hindering is entirely negligible, as found in Section 3.2.2.

This analysis is based solely on statistical inference with the F-test, which is predicated on the assumption of normally distributed residuals. All the free parameters are treated on equal footing, and the presence or absence of hindering is inferred from the statistical significance of the improvement in adding the parameter a_1 to the modeling. In contrast, the approach used in the body of the text (Section 3) utilizes the specifics of hindering and is more direct. It invokes statistical inference with the non-parametric MK-test only to deduce the presence of hindering directly from the data, without any modeling and without any assumptions about normally distributed populations. It is gratifying that two independent, entirely different approaches reach the same conclusions for all the cases studied here.

References

Acemoglu, D., 2008. Introduction to Modern Economic Growth. Princeton University Press.
Acemoglu, D., Johnson, S., Robinson, J.A., 2005. Chapter 6: Institutions as a fundamental cause of long-run growth. In: Handbook of Economic Growth, vol. 1. Elsevier, pp. 385–472.
Akaike, H., 1969. Fitting autoregressive models for prediction. Ann. Inst. Stat. Math. 21, 243–247.
Becker, G., Murphy, K., Tamura, R., 1990. Human capital, fertility, and economic growth. J. Polit. Econ. 98, S12–37.
Draper, N.R., Smith, H., 1998. Applied Regression Analysis, (3rd ed.) John Wiley & Sons.
Elitzur, M., 1992. Astronomical Masers. Dordrecht: Kluwer Academic Publishers.
Elitzur, M., 1995. Masers in the sky. Sci. Am. 272, 68–74.
Griliches, Z., 1957. Hybrid corn: an exploration in the economics of technological change. Econometrica 25, 501–522.
Johnston, L., Williamson, S.H., 2019. What Was the U.S. GDP Then?. MeasuringWorth. http://www.measuringworth.com/usgdp/
Jones, C.I., 1995. Time series tests of endogenous growth models. Q. J. Econ. 110, 495–525.
Jones, C.I., 2016. Chapter 1: The facts of economic growth. In: Handbook of Macroeconomics, 2. Elsevier, pp. 3–69.
Kingsland, S., 1982. The refractory model: the logistic curve and the history of population ecology. Q. Rev. Biol. 57, 29–52.
Kocsis, T., Kovács-Székely, I., Anda, A., 2017. Comparison of parametric and non-parametric time-series–analysis methods on a long-term meteorological data set. Cent. Euro. Geol. 60, 316–332.
Levin, S.A., 1976. Population dynamic models in heterogeneous environments. Ann. Rev. Ecol. Syst. 7, 287–310.
Maddala, G.S., Lahiri, K., 2009. Introduction to Econometrics, (4th ed.) Chichester: Wiley.
Mankiw, N.G., Romer, D., Weil, D.N., 1992. A contribution to the empirics of economic growth. Q. J. Econ. 107, 407–437.
O'Neill, B.C., Balk, D., Brickman, M., Ezra, M., 2001. A guide to global population projections. Demogr. Res. 4, 203–288.
Rossi-Hansberg, E., 2019. Geography of Growth and Development. Oxford Research Encyclopedia of Economics and Finance. Oxford University Press USA.
Solow, R.M., 1956. A contribution to the theory of economic growth. Q. J. Econ. 70, 65–94.
Stock, J.H., Watson, M.W., 2007. Why has US inflation become harder to forecast? J. Money Credit Bank. 39, 13–33.
Thomas, R., Williamson, S.H., 2019. What was the U.K. GDP then?. MeasuringWorth. http://www.measuringworth.com/ukgdp/.